Verification of a fully kinetic ion model for electromagnetic simulations of high-frequency waves in toroidal geometry

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ABSTRACT
For the study of high-frequency electromagnetic waves in tokamaks, an electromagnetic simulation model, in which the ion dynamics is described by a six-dimensional Vlasov equation and the electron dynamics is described by a drift kinetic equation, is formulated and implemented in the global gyrokinetic toroidal code (GTC). Analytic dispersion relations are derived in reduced systems and compared with various theories to verify the model. Linear simulations of a generalized ion Bernstein wave and ion cyclotron emission are verified by comparing the GTC simulation results with analytic dispersion relation theory and magnetoacoustic cyclotron instability theory, respectively, in cylindrical geometry.

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I. INTRODUCTION
There are a variety of high-frequency waves excited by energetic particles in fusion plasmas, such as compression Alfvén eigenmodes (CAE), global Alfvén eigenmodes (GAE), and ion cyclotron emission (ICE), recently observed in many tokamaks.1–10 Many new phenomena arising from high-frequency modes need more experimental, theoretical, and simulation research to understand their effects on plasma confinement and their potential roles as diagnostic tools for α-particles in burning plasmas.

CAEs and GAEs typically have a frequency close to the ion gyrofrequency. They can be driven by super-Alfvénic energetic particles and are frequently observed in the NSTX-U and DIII-D tokamak experiments with low magnetic fields. The 3D hybrid MHD-kinetic code HYM,11 in which fast ions are simulated using the particle-in-cell (PIC) method and full-orbit equations of motion, has been used in simulations of CAE and GAE excitation and stabilization in the NSTX-U.12,13 ICES, on the other hand, are usually excited due to magnetoacoustic cyclotron instability (MCI), in which the compression Alfvén wave frequency is close to the harmonics of the energetic ion gyrofrequency. The basis for diagnostic exploitation of the ICE has been greatly strengthened by recent nonlinear simulations using fully kinetic ions and fluid electrons in a slab geometry.14 Such simulations have confirmed that ICE intensity scales linearly with concentration of energetic ions ($n_f/n_0$, where $n_f$ is the energetic ion density and $n_0$ is the electron density), which agrees with the measurement in the JET tokamak.15–17 Therefore, ICE potentially offers a unique diagnostic tool for the α-particles produced by a fusion reaction and is planned to be used in future deuterium-tritium experiments in the ITER tokamak.18 These local simulations of the ICE in a simplified geometry, however, cannot address some key physics, such as mode structure, the time evolution, and the dependence on the fast-ion distributions, as observed in the DIII-D experiments.9

To simulate high-frequency waves whose frequency is close to or higher than the ion cyclotron frequency, a fully kinetic ion model is needed to resolve the ion cyclotron motion. An electromagnetic, fully kinetic PIC simulation model has been used in EPOCH19 and iPIC3D20 simulations of the ICE21 and magnetic reconnection,20 respectively, using Cartesian coordinates for slab geometry. In space plasma simulations, there are also several electromagnetic models in which ions are fully kinetic, such as GeFi22 and AIKEF.23 In tokamak plasma simulations, however, accurate description of the toroidal geometry is essential for the study of resonant excitation and mode structures, which is absent in the models mentioned above. The kinetic-MHD toroidal codes HYM and M3D-K24 use an MHD model for thermal plasmas and a gyrokinetic model for energetic particles to describe the macroscopic behavior of plasmas, although the kinetic effects of thermal plasmas can in principle be incorporated. In previous gyrokinetic toroidal code (GTC)25 simulations, the Boris push has been implemented for integrating the full particle orbit in cylindrical26 and toroidal geometries.27 In the electrostatic simulation using fully kinetic ions, a linear...
dispersion relation and nonlinear particle trapping have been verified for a lower hybrid wave (LHW)\cite{28} and an electrostatic ion Bernstein wave (IBW).\cite{29} The frequency and mode structure in simulations agree well with those of theoretical prediction.\cite{30} GTC simulations using fluid ions and drift kinetic electrons have also been carried to study the LHW linear mode conversion and absorption in the toroidal geometry.\cite{31}

In this work, we extend the GTC simulation in the toroidal geometry to the electromagnetic simulation model, in which the ion dynamics is described by a six-dimensional Vlasov equation and the electron dynamics is described by a drift kinetic equation. An analytical dispersion relation is derived in two reduced systems with massless fluid electrons. We implement the model in the GTC code and carry out linear simulations of waves propagating in the perpendicular (to an equilibrium magnetic field) direction, in which the ion response can be neglected. In these simulations, the compressional Alfvén wave and the generalized IBW\cite{32} frequency are verified using an analytical dispersion relation. Then, simulations of both thermal and energetic ions are carried out for the ICE excitation. Consistent with analytic theory,\cite{33} GTC simulation results show that the ICE linear growth rate is approximately proportional to the square-root of the concentration of the energetic ions \( (n_i/n_0) \), and that different cyclotron harmonics have comparable growth rates.

The remainder of this paper is organized as follows. Section II describes the formulation of the kinetic simulation model for high-frequency waves in toroidal geometry. The analytic dispersion relations in reduced systems are derived and compared with a fully kinetic model in Sec. III. The verification of GTC simulations of the generalized IBW dispersion relation and the excitation of ICE is presented in Sec. IV. Finally, Sec. V presents the conclusion and discussion.

II. SIMULATION MODEL

To study high-frequency electromagnetic waves in a tokamak, we develop a new electromagnetic simulation model with fully kinetic ions and drift kinetic electrons. In Sec. II A, we use a Vlasov equation in six-dimensional phase space to describe the ion dynamics. In Sec. II B, we employ a drift kinetic equation to describe the electron dynamics. Then, field equations, including Poisson’s equation, the parallel Ampère’s law, and the electron perpendicular force balance equation are used to close the system in Sec. II C.

A. Fully kinetic ions

For fully kinetic ions, the dynamics is governed by a Vlasov equation in six-dimensional phase space:

\[
L_{fi}(r,v,t) = 0.
\]

Here \( r, v, t \) are the particle position, velocity, and time, respectively; \( f_i \) is an ion distribution function; and \( L \) is a nonlinear propagator, which can be decomposed into an equilibrium part \( L_0 \) and a perturbed part \( \delta L, L = L_0 + \delta L \):

\[
L_0 = \frac{\partial}{\partial t} + v \cdot \nabla + \frac{q_i}{m_i} \frac{1}{c} \cdot \frac{\nabla}{\times} \mathbf{B}_0 \cdot \nabla, \tag{2}
\]

\[
\delta L = \frac{q_i}{m_i} \left( \frac{\nabla}{\times} \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) \cdot \nabla, \tag{3}
\]

where \( q_i \) and \( m_i \) are the ion charge and ion mass, \( \mathbf{B}_0 \) is the equilibrium magnetic field, and \( \mathbf{E} \) and \( \mathbf{B} \) are the perturbed electric field and perturbed magnetic field, respectively. The ion distribution function is also decomposed into an equilibrium part and perturbed part as \( f = f_0 + \delta f_i \). The equilibrium distribution \( f_0 \) obeys

\[
L_{f0} = 0. \tag{4}
\]

The solution of this equation is any distribution function of constant of motion, e.g., uniform isotropic Maxwellian and modified slowing down distribution used in this work. Defining the particle weight as \( w_i = \delta f_i / f_0 \), we obtain the weight evolution equation:

\[
\frac{dw_i}{dt} = -\frac{1 - w_i}{f_0} \delta L_{f0}. \tag{5}
\]

The ion motion in the phase space under self-consistent electromagnetic fields is given by

\[
\frac{dv_i}{dt} = \frac{q_i}{m_i} \left( \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right), \tag{6}
\]

\[
\frac{dr_i}{dt} = \mathbf{v}. \tag{7}
\]

We use the Boris scheme in magnetic coordinates to push ions in the GTC code.\cite{34} The perturbed number density \( n_i \) and current density \( j_i \) can be obtained from the velocity moments of \( \delta f_i \), i.e.,

\[
\delta n_i = \int \delta f_i dv, \tag{8}
\]

\[
\delta j_i = q_i \int v \delta f_i dv. \tag{9}
\]

Here, the particle charge and current are deposited onto the surrounding eight grid points in the 3D space using linear interpolation.\cite{35}

B. Drift kinetic electrons

The electrons are described by the drift kinetic equations.\cite{36}

Using the gyrocenter position \( \mathbf{X} \), parallel velocity \( v_x \), and magnetic moment \( \mu \) as independent variables in five-dimensional phase space, the drift kinetic equation is

\[
L_{fe}(\mathbf{X}, v_x, \mu, t) = 0, \tag{10}
\]

where \( f_e \) is the electron distribution function and \( L_e \) is the gyrocenter propagator in symplectic form.\cite{37} \( L_e \) can be decomposed into an equilibrium part \( L_{e0} \), a first-order linear part \( \delta L_{e1} \), and a second-order nonlinear part \( \delta L_{e2} \) as \( L_{e0} + \delta L_{e1} + \delta L_{e2} \):

\[
L_{e0} = \frac{\partial}{\partial t} + \left( \frac{v_x}{B_0} \mathbf{B}_0^* + \frac{c \mathbf{v}_x}{q_e B_0} \times \mu \mathbf{V} \mathbf{B}_0 \right) \cdot \nabla - \frac{\mu}{m_e B_0} \mathbf{B}_0^* \cdot \mathbf{V} \mathbf{B}_0 \frac{\partial}{\partial v_x}, \tag{11}
\]

\[
\delta L_{e1} = \left( \frac{v_x}{B_0} \mathbf{B}_0^* + \frac{c \mathbf{v}_x}{q_e B_0} \times \mu \mathbf{V} \mathbf{B}_0 \right) \cdot \nabla + \left( -\frac{\mu}{m_e B_0} \mathbf{B}_0^* \cdot \mathbf{V} \mathbf{B}_0 - \frac{B_0^*}{m_e B_0} \left( q_e \mathbf{V} \delta \phi + \mu \mathbf{V} \delta B_0 \right) \right) \cdot \frac{\partial}{\partial v_x}, \tag{12}
\]

\[
\delta L_{e2} = -\frac{B_0^*}{m_e B_0} \cdot \left( q_e \mathbf{V} \delta \phi + \mu \mathbf{V} \delta B_0 \right) \frac{\partial}{\partial v_x}. \tag{13}
\]
where $B' = B_0^\parallel + \delta B_\perp$, $B_0^\parallel = B_0$, and $\delta B_\perp = V \times b_0$.

The electron distribution function is decomposed into an equilibrium part and a perturbed part as $f_e = f_{eq0} + \delta f_e$. The equilibrium distribution function satisfies

$$L_{eq} f_{eq0} = 0.$$  \hspace{1cm} (14)

$f_{eq0}$ is approximated as a Maxwellian distribution function by neglecting electron finite Larmor radius effects and equilibrium parallel flow.

The equations of particle motion corresponding to Eq. (10) are given in Appendix A of Bao’s paper.\textsuperscript{37}

The perturbed electric field is defined by the perturbed scalar and vector potential:

$$\nabla \phi = \frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi + \frac{q_e n_0}{m_e} \mathbf{B}_0 \cdot \mathbf{v} \nabla \phi + \frac{q_i n_0}{m_i} \frac{\partial \phi}{\partial t} + \frac{\mathbf{b}_0}{m_e} \cdot \nabla \phi.$$  \hspace{1cm} (18)

C. Field equations

In the particle simulation, the plasma is treated as a set of computational particles interacting with each other through self-generated electromagnetic fields. To close the system, therefore, we need to solve the field equations. The electrostatic is solved by the Poisson’s equation for fully kinetic ions and drift kinetic electrons:\textsuperscript{35}

$$\nabla^2 \phi = \frac{1}{\epsilon} \left( \frac{\partial^2 \phi}{\partial t^2} + \mathbf{v} \cdot \nabla \phi \right) - \frac{\partial \phi}{\partial t} = -\frac{4\pi}{\epsilon} \left( \mathbf{b}_0 \cdot \mathbf{v} E + \frac{m_i}{m_e} \mathbf{b}_0 \mathbf{v} \cdot \nabla \mathbf{b}_0 \right).$$  \hspace{1cm} (19)

Finally, the compressible magnetic perturbation is solved from the perpendicular electron force balance equation:\textsuperscript{36}

$$\nabla^2 \mathbf{B}_\perp = \frac{4\pi}{B_0} \left( n_0 q_i - n_0 q_e \frac{\partial \mathbf{U}_e}{\partial t} \right).$$  \hspace{1cm} (20)

Here, the effective potential $\chi$ for the perturbed Lorentz force is defined\textsuperscript{36} as

$$\nabla^2 \chi = -\frac{1}{n_0 q_e} \left( \frac{m_e}{m_i} \mathbf{v} E + \frac{m_i}{m_e} \mathbf{b}_0 \right).$$  \hspace{1cm} (22)

The perturbed electric field is defined by the perturbed scalar and vector potential:

$$\nabla^2 \mathbf{A}_\perp = -\nabla \phi \mathbf{B}_0 + \frac{\partial \mathbf{U}_e}{\partial t}.$$  \hspace{1cm} (23)

Now, we complete the formulation of the fully kinetic ion model for electromagnetic simulations. These field equations are solved by GTC in a field-aligned mesh\textsuperscript{36} using either finite difference method\textsuperscript{35} or the finite element method.\textsuperscript{37}

III. ANALYTIC DISPERSION RELATION IN REDUCED SYSTEMS

In order to verify the simulation model given in Sec. II, we derive a corresponding linear dispersion for two reduced systems. Here, we
consider uniform ambient magnetic field and uniform equilibrium plasma density and temperature profiles. Furthermore, we reduce the drift kinetic electrons to massless cold fluid electrons, retaining only the linear terms. Then, Eqs. (17), (18), and (21) reduce, respectively, to

\[
\frac{\partial \delta n_e}{\partial t} = -n_b \mathbf{b}_y \cdot \mathbf{V} \delta u_{||},
\]

\[
\frac{\partial \delta A_i}{\partial t} = -c \mathbf{b}_y \cdot \mathbf{V} \delta \phi,
\]

\[
\delta b_y = \frac{4\pi n_0 q_e}{B_0} (\chi - \delta \phi).
\]

Using the quasi-neutrality condition, Poisson’s equation [Eq. (19)] reduces to

\[
\delta \phi = \frac{B_0^2}{4\pi n_0 q_e^2} (q_i \delta n_i + q_e \delta n_e) + \chi.
\]

Since electrons move much faster than ions in the parallel direction, we can neglect the perturbed parallel ion current. Then, from the parallel Ampère’s law, we have

\[
\delta u_{||} = \frac{c}{4\pi n_0} \nabla_\perp \delta A_{||}.
\]

A. Two fluid, cold plasmas

In the first reduced system, we treat ions as cold fluid plasmas and \( q_i = -q_e = e \). The ion continuity equation is

\[
\frac{\partial \delta n_i}{\partial t} + n_0 \mathbf{V}_i \cdot \delta \mathbf{u}_{\perp} = 0.
\]

Using the ion dynamic equation and normal mode ansatz, \( \omega = -i \omega_0 \), \( \mathbf{V} = \mathbf{k} \), we easily derive the following equation:

\[
\delta \mathbf{u}_{\perp} = \left( \frac{i \omega_0 q_i}{m_i (\omega_0^2 - \Omega_i^2)} \right) \left( \delta \mathbf{E}_{\perp} + \frac{i \Omega_i}{\omega_0} \delta \mathbf{E}_{\perp} \times \mathbf{b}_0 \right).
\]

Combining the definition of perpendicular ion current,

\[
\delta j_{||} = n_0 q_i \delta \mathbf{u}_{\perp},
\]

together with Eqs. (22)–(31), and using the non-trivial solution condition, we derive the dispersion relation as

\[
(S - n_{\perp}^2)(S' - n_{\perp}^2) - D^2 = 0,
\]

where

\[
S' = \frac{\omega_p^2}{\Omega_i^2} \frac{1}{1 - (\omega_0/\Omega_i)^2} \approx S - 1,
\]

\[
D' = -\frac{\omega_p^2}{\Omega_i^2} + \frac{\omega_p^2 \Omega_i}{\omega_0 (\omega_0^2 - \Omega_i^2)} \approx D.
\]

The difference between \( S' \) and \( S \) comes from the fact that we have dropped the displacement current in our model. In Eq. (33), the factor \( S' - n_{\perp}^2 \) corresponds to \( S - n^2 \) in the ideal MHD theory. The difference comes from the fact that we assume \( k_\parallel \ll k_\perp \) in the parallel Ampère’s law and the perpendicular force balance equation, and we also drop some coupling terms between the parallel and perpendicular wave vectors.

When we derive the parallel dispersion relation, therefore, we use the assumption \( n_{\perp} \gg n_i \) instead of letting \( n_{\perp} = 0 \),

\[
\frac{D^2}{n_{\perp}^2} = \frac{y^2 S^2}{n_i^2} - S \approx \frac{y^2 n_i^2}{n_{\perp}^2} \sim 0,
\]

where \( y = \frac{\omega_0}{\Omega_i} \). Furthermore, we have

\[
k_{||}^2 v_A^2 / \Omega_i^2 = y^2 / (1 - y^2),
\]

which is the dispersion relation of a shear Alfvén wave modified by an ion cyclotron frequency.

Considering the perpendicular dispersion relation, we let \( n_{\perp} = 0 \) directly and easily obtain

\[
k_{||}^2 v_A^2 / \Omega_i^2 = y^2,
\]

which is a compressional Alfvén wave (CAW).

B. Fully kinetic ions and massless fluid electrons

If we retain only the leading order linear terms for the fully kinetic ion model, then Eq. (5) reduces to

\[
\frac{d \omega_i}{d t} = -\frac{1}{f_{\text{io}}} \delta \mathbf{f}_{\text{io}}.
\]

Using the unperturbed particle orbit integration method, combining Eqs. (2), (3), (8), and (9), and using a uniform Maxwellian distribution function

\[
f_{\text{io}} = n_0 \left( \frac{m_i}{2\pi T_i} \right)^{3/2} \exp \left( -\frac{m_i}{2T_i} \frac{v_x^2 + v_y^2}{2} \right),
\]

we derive the ion density:

\[
\delta n_i = F \left[ \frac{n_i^2}{k_{||} \Omega_i} \right] \delta E_x + F \left[ \frac{n_i^2}{k_{||} \Omega_i} \right] \delta E_y + F \left[ \frac{n_i^2}{k_{||} \Omega_i} \right] \delta E_z,
\]

and the ion current:

\[
\delta j_i = \sigma \cdot \delta \mathbf{E}.
\]

The conductivity tensor can be written as
The electrostatic IBW is recovered exactly when we retain only 
the first kind modified Bessel function of order \( l \) and the argument of \( J_l \) is 
\( \zeta = k_{\perp}v_{\perp}/\Omega_i \).

Now, using Eqs. (22)–(29), (39) and (40), we derive the dispersion relation in which waves propagate perpendicular and parallel to \( B_0 \), respectively. In the perpendicular direction, the dispersion relation is

\[
\frac{b^2}{0.5\beta_i} + F_1 = \frac{b^2}{0.5\beta_i} + \left( \frac{F_3}{0.5\beta_i} \right) (1 + F_2) = 0, \quad (41)
\]

where

\[
b = \frac{k_{\perp}v_{\parallel}}{\Omega_i} = k_{\perp}\rho_i;
\]

\[
\beta_i = 2\frac{v_{\parallel}}{v_A} = \frac{8\pi n_i T_i}{B_0^2},
\]

\[
F_1 = \sum_{l=0}^{\infty} \left( \frac{2(\delta(l))}{\gamma^2 - \beta^2} \right) \exp(-b^2) \left( \left( \frac{l^2}{\beta^2} + 2(l^2 - \beta^2) \right) J_l - 2b^2 J_{l+1} \right),
\]

\[
F_2 = \sum_{l=1}^{\infty} \frac{2l}{\gamma^2 - \beta^2} \exp(-b^2) \left( \left( \frac{l^2}{\beta^2} - 1 \right) J_l + 2b^2 J_{l+1} \right),
\]

\[
F_3 = \sum_{l=0}^{\infty} \frac{4l}{\gamma^2 - \beta^2} \exp(-b^2) J_l,
\]

\[
\beta_i = \sum_{l=0}^{\infty} \left( \frac{4 - 2\delta(l)}{\gamma^2 - \beta^2} \right) \exp(-b^2) (1 - b^2) J_l + b^2 J_{l+1}.
\]

\( v_{\parallel} = \sqrt{T_i/m_i} \) and \( v_A = \sqrt{B_0^2/(4\pi n_i m_i)} \) are, respectively, the ion thermal speed and Alfvén speed; \( \delta \) is the Dirac delta function; \( l \) the first kind modified Bessel function of order \( l \); and the argument of \( J_l \) is \( b^2 \). If the ion temperature is very low, we have \( \beta_i \ll 1 \), \( \beta_i \ll 1 \). Then, using the asymptotic formula \( I_l(b^2) \approx \frac{1}{\sqrt{\pi}} \left( \frac{b^2}{2} \right) \), we retain only the lowest-order terms, and Eq. (41) reduces to

\[
\left( \frac{b^2}{0.5\beta_i} \right) + \frac{b^2}{0.5\beta_i} \cdot \frac{1}{\gamma^2 - \beta^2} = 0.
\]

Then, \( \gamma^2 = 2b^2/\beta_i \) is equivalent to \( \omega = k_{\perp}v_{\perp} \), which corresponds to a compressional Alfvén wave in fluid theory. We plot the dispersion relation numerically according to Eq. (41), as shown in Fig. 1 for parameter \( \beta_i = 0.1 \).

The wave described by Eq. (41) is a generalized electromagnetic IBW (ion Bernstein wave, blue solid line). It becomes a compressional Alfvén wave in the long wavelength limit, and an electrostatic IBW in the short wavelength limit. \( \delta B_0 \), coming from the polarization currents, provides a perpendicular inductive electric field, which causes an electron density perturbation via the \( \delta E \times B_0 \) drift effect, as shown in Poisson equation Eq. (28). The \( (1 + F_2) \) term in the dispersion relation comes from \( \delta B_0 \), which is much less important to the dispersion relation in the short wavelength limit. Therefore, \( F_3 = 0 \) is approximately satisfied, which is very close to the electrostatic IBW dispersion relation. In fact, in the short wavelength limit, the quasi-neutrality condition fails. The electrostatic IBW is recovered exactly when we retain \( \nabla^2 \delta \phi \) in Poisson’s equation [Eq. (19)].

In the parallel direction, assuming \( \zeta = k_{\parallel}v_{\parallel}/\Omega_i \ll 1 \), we have

\[
J_n(\zeta) = \begin{cases} 1, & n = 0, \\ \pm \zeta/2, & n = \pm 1, \\ 0, & \text{others}, \\ \end{cases} J_n'(\zeta) = \begin{cases} -\zeta/2, & n = 0, \\ \pm 1/2, & n = \pm 1, \\ 0, & \text{others}. \end{cases}
\]

Therefore, the dispersion relation can be written as

\[ \omega = k_{\parallel}v_{\parallel} \]
Similarly, at very low ion temperatures, $\zeta \ll 1$, and only the lowest-order terms are retained. Equation (43) now reduces to

$$\left( \frac{b^2}{0.5\beta_i} - \frac{y^2}{1 - y^2} \right) - \frac{y^2}{2y_{\text{thi}}} + \frac{y^2}{1 - y^2} = 0,$$

which is exactly the same as Eq. (33) from the fluid theory. The kinetic dispersion relation $b = \frac{\omega_i}{\sqrt{2}}$, $\beta_i = 0.02$ is shown in Fig. 2. Compared with the MHD theory, the kinetic dispersion relation has a lower real frequency and a Landau damping effect.

IV. VERIFICATION OF ICE SIMULATION IN PREPENDICULAR DIRECTION

In this section, we verify the simulation results for the generalized IBW dispersion relation and the excitation of ICE using analytic theory and simple geometry. The model described in Sec. II is implemented in the GTC. The simulations are performed in cylindrical geometry with a uniform magnetic field and uniform equilibrium profile.

We first verify the simulation results of the generalized IBW dispersion relation. We carry out an initial value simulation with parameters $r_0 = 0.3164a$, $r_1 = 0.3465a$, $\Delta t = 0.0558Q_\Omega^{-1}$, and $\omega_{\text{thi}} = 10.05Q_\Omega$. Here $r_0$ is the inner boundary, $r_1$ is the outer boundary, $a$ is the minor radius, and $\Delta t$ is the time step. Moreover, we select only the poloidal harmonic $m = 0$ of the perturbed fields in every time step to mimic slab geometry for verification. In the first simulation, we begin from a random perturbed density and retain modes in the full range of the $k_\perp$ spectrum. The initial random perturbations decay after 18 ion cyclotron periods. The spectral signals are shown in Fig. 3, where the background color represents the mode amplitudes. The analytic dispersion relation is also plotted in Fig. 3 as solid blue lines, which agree very well with the strongest spectral signals from the simulations. In the next simulation, we give an initial perturbed density profile $\frac{dn}{m} = 10^{-5}\sin(6 \times 2\pi \frac{r - r_0}{r_1 - r_0})$ and filter the field at every time step to allow only a single mode with a $k_\perp = 12\pi/\Delta r$ component. By scanning the value of $\rho_i$, we obtain the dispersion relation for various $k_\perp \rho_i$, shown in Fig. 3 as blue circles.

![FIG. 3. The dispersion relation. GTC simulation results, keeping all $k_\perp$ modes and only a single $k_\parallel$ mode, are represented by the color and blue circles, respectively; solid lines are the analytic kinetic solution.](image-url)
which agree with the analytic solution. These simulation results provide a solid verification of the physical and numerical models for the ICE simulation in the perpendicular direction. Notably, the signals, especially for high cyclotron harmonics, are very weak when \(k_{\perp} \rho_i\) is very small. It might come from the cyclotron damping effect of bulk ions.

Next, the simulation results for the ICE excited by energetic particles are verified by comparing them with Dendy’s analytic theory, which holds the view that ICE arises from a magnetoacoustic cyclotron instability (MCI). The MCI is driven by free energy in the phase space of energetic particles, whose distribution in velocity space exhibits a natural population inversion. The \(\pi\)-particles produced by fusion reaction have, at birth, an anisotropic shell distribution in the velocity space. The fast ions produced by the neutral beam injection (NBI) have an anisotropic shell distribution at birth. However, the shell distribution will relax to the slowing-down distribution at the collisional timescale, which is much longer than the MCI excitation time. Therefore, the fast ion distribution with the inverted population that excites the MCI can be somewhere between the shell distribution and slowing-down distribution. An isotropic (spherical) shell distribution has been used in the analytic theory for the MCI excitation. Effects of the spherical shell distribution with a finite thickness and the ring-beam distribution with a perpendicular velocity spread have been shown to affect the excitation and evolution of the MCI in the nonlinear full-\(f\) simulations. These physical effects will be further studied in our future nonlinear full-\(f\) simulations using the new simulation models described in Sec. II. For the linear \(\delta f\) simulation reported in this paper for the verification of our simulation model, we construct two isotropic distribution functions based on a slowing-down distribution function that has already been implemented in the GTC:

\[
F_{\delta f}(v) = n_f \frac{v^6 \mathcal{H}(v_\theta - v)}{C_6 \, v^3 + v_c^3},
\]

and

\[
F_{\delta f}(v) = n_f \frac{v^4 \mathcal{H}(v_\theta - v)}{C_4 \, v^3 + v_c^3},
\]

where \(\mathcal{H}\) is Heaviside function, \(v_\theta = 1.5 \sqrt{T_f/m_f}\), \(v_c = 2v_\theta\), \(C_6 = \int_0^{\infty} \pi v^6 dv = \frac{\pi}{2} (v^6_c - 2v^4_c v_\theta^2 + 2v^2_c \ln(1 + v^2_c/v_\theta^2))\), and \(C_4 = \frac{1}{12} \pi \int_0^{\infty} \pi v^4 dv = \frac{\pi}{2} v^4_c - 4\pi v_\theta v_c^2 + \frac{1}{2} \pi v^2_c (\sqrt{3\pi} - 6\sqrt{3} \arctan(\frac{v_c}{\sqrt{v_\theta^2}}) + 6\ln(v_\theta + v_c) - 3 \ln(v^2_c - v_\theta v_c + v_c^2))\). We also use parameters similar to those in the analytic theory: an energetic proton population in deuterium plasmas with parameter values \(B_0 = 3.1\ T, T_i = 1\ keV, T_f = 3\ MeV,\) and \(n_0 = 2 \times 10^{19}\ \text{cm}^{-3}\). Other parameters are \(r_0 = 0.3164, a, r_1 = 0.3465, a, a = 0.4242\), and \((r_1 - r_0)/\rho_i = 1.227\). The initial perturbed density profile is \(n_f/n_0 = 10^{-3} \sin(32\pi r_i/r_0)\) and only the \(k_{\perp} = 3\pi/(r_1 - r_0) = 0.0819\) component is retained in the simulation. Using these parameters, the linear simulations find the MCI instability.

To ensure that the simulation results are physical, we have verified the conservation of the equilibrium particle kinetic energy as a check for the accuracy of the time integration. More importantly, we have performed numerical convergences for the number of particles per cell and the size of time step size as shown in Fig. 4. The mode \(\omega = 4\Omega_i = 2\Omega\) is retained and the \(F_{\delta f}\) distribution function is used in these simulations. Considering the converged simulation results, we use 2048 grids in the radial direction, approximately average 6400 thermal ions and 640 fast ions per radial grid, and a time step of \(\Delta t = 2 \times 10^{-4}\Omega_i^{-1}\) to obtain accurate results in all subsequent simulations. Now we scan the value of \(n_f/n_0\) to obtain the relation between growth rate and concentration of energetic ions, which is shown in Fig. 5. The GTC simulation results of the growth rate for the \(\omega = 2\Omega\) 

![Image](image-url)

**FIG. 4.** Numerical convergence studies showing the dependence of the MCI growth rate on: (a) the number of fast ions per radial grid \(n_f\); and (b) the size of the time step \(\Delta t\).

![Image](image-url)

**FIG. 5.** Dependence of the ICE growth rate on the concentration of energetic ions. The red solid line is from Dendy’s analytic theory. Orange points and blue squares are from the GTC simulation using slightly different distribution functions.
mode are plotted as orange points for $F_{F6}$ and blue squares for $F_{F8}$. Their fitted straight lines (the orange dashed line and the red solid line) show that the growth rate scales linearly as $\sqrt{n_f/n_0}$, which agrees with the analytic theory and other PIC simulations.\(^{14,39}\)

However, there is a difference between the slope in our simulation using $F_{F6}$ and the slope using analytic theory. Our results using $F_{F4}$ have a similar slope to that of the analytic theory. The $F_{F6}$ distribution has a greater inverse population than the $F_{F4}$ distribution and a higher effective temperature than the $T_f = 3$ MeV used in the analytic theory; and therefore, the orange line has a steeper slope. It is worth noting that, when $n_f/n_0$ is very small ($\sqrt{n_f/n_0} \approx 0.01$), the growth rates from the simulations do not follow this linear scaling, implying the existence of bulk ion cyclotron damping effects and confirming previous results.\(^{14}\)

Selecting different $k_x \rho_i$ modes in the simulation allows the examination of various resonant frequencies: $\omega = 1, 2, 3, 4, 5 \Omega_i$. As shown in Fig. 6, linear growth rates for different harmonics of the ICE are obtained from the fully kinetic simulations using $F_{F6}$ for the three values $n_f/n_0 = 0.5\%, 1\%, 2\%$. The results show that all harmonics have similar growth rates for each concentration of energetic ions.

In future work, we will explore the MCI in relation to the ICE excitation in fusion plasmas in toroidal geometry using nonlinear simulations, comparing the results with other nonlinear 2D simulations.\(^{37,40}\) We will then validate our model by comparing simulation results with experimental measurement results to study the ICE mode structure, saturation, and amplitude modulation correlated with edge-localized mode (ELM) activity.\(^{11}\)

**V. CONCLUSIONS AND DISCUSSION**

A new electromagnetic simulation model, in which ion dynamics is described by a six-dimensional Vlasov equation and electron dynamics is described by a drift kinetic equation or massless fluid equation, was formulated and implemented in the GTC code for simulations of high-frequency waves in the toroidal geometry. Linear dispersion relations of such high-frequency waves in both perpendicular and parallel directions were verified. Using a model with fully kinetic ions and massless fluid electrons, we derived the general IBW dispersion relation in the perpendicular direction and kinetic shear Alfvén wave in the parallel direction, which confirms that our simulation model faithfully preserves the high-frequency waves. Furthermore, the simulation results of the ICE excitation in the cylindrical geometry were verified by comparing them with analytic theory. The simulation results recovered the important scaling of the MCI growth rate with the fast ion concentration.

Furthermore, the nonlinear dynamics of high-frequency waves (ICE, CAE, and GAE waves) will be studied using the new simulation capability.

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**AUTHOR DECLARATIONS**

**Conflict of Interest**

The authors have no conflicts to disclose.

**Author Contributions**

**Yangyang Yu:** Conceptualization (equal); Data curation (lead); Formal analysis (lead); Investigation (lead); Methodology (lead); Project administration (lead); Software (supporting); Validation (lead); Visualization (lead); Writing – original draft (lead); Writing – review and editing (equal).

**Xishuo Wei:** Writing – original draft (supporting); Writing – review and editing (supporting).

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**Zhihong Lin:** Conceptualization (equal); Formal analysis (supporting); Funding acquisition (lead); Investigation (supporting); Methodology (supporting); Resources (lead); Software (lead); Supervision (lead); Writing – original draft (supporting); Writing – review and editing (equal).

**DATA AVAILABILITY**

The data that support the findings of this study are available from the corresponding author upon reasonable request.

**REFERENCES**
