1 Gyrokinetic Poisson equation

The gyrokinetic Poisson equation

$$\phi - \tilde{\phi} = \frac{T_i}{n_i Z_i^2} \left(Z_i n_i - e n_e \right) \tag{1}$$

Electrostatic potential and densities are decomposed into zonal and non-zonal parts

$$\phi = \langle \phi \rangle + \delta\phi$$
$$n = \langle n \rangle + \delta n$$

with

$$\langle \delta\phi \rangle = 0, \qquad \langle \delta n \rangle = 0$$

Decomposing potential and densities into zonal and non-zonal parts and subtracting flux-surface averaged Eq. (1) from the total one

$$\delta\phi - \widetilde{\delta\phi} + \langle \tilde{\phi} \rangle - \widetilde{\langle \phi \rangle} = \frac{T_i}{n_i Z_i^2} \left(Z_i \delta n_i - e \delta n_e \right) \tag{2}$$

or in the long-wavelength limit

$$\delta\phi - \widetilde{\delta\phi} + \left(\langle k_\perp^2 \rho_i^2 \rangle - k_\perp^2 \rho_i^2 \right) \langle \phi \rangle + \langle k_\perp^2 \rho_i^2 \delta\phi \rangle = \frac{T_i}{n_i Z_i^2} \left(Z_i \delta n_i - e \delta n_e \right). \tag{3}$$

Note, that the terms $\left(\langle k_\perp^2 \rho_i^2 \rangle - k_\perp^2 \rho_i^2 \right) \langle \phi \rangle$ and $\langle k_\perp^2 \rho_i^2 \delta\phi \rangle$ represent the coupling between magnetic field and $\phi_{n=0,m\neq0}$ harmonics. Without these coupling Eq. (3) becomes

$$\delta\phi - \widetilde{\delta\phi} = \frac{T_i}{n_i Z_i^2} \left(Z_i \delta n_i - e \delta n_e \right). \tag{4}$$

2 Zonal flow equation

The flux-surface averaged gyrokinetic Poisson equation using the Padé approximation,

$$\langle \nabla_\perp^2 \phi \rangle = \langle \left[\frac{1}{\rho_i^2} \frac{T_i}{n_i Z_i^2} - \frac{T_i}{n_i Z_i^2} \nabla_\perp^2 \right] \left(e n_e - Z_i n_i \right) \rangle. \tag{5}$$

The Laplacian

$$\nabla^2 = \frac{1}{J} \partial_\alpha J g^{\alpha\beta} \partial_\beta \tag{6}$$

The flux-surface averaging

$$\langle \phi \rangle \equiv \frac{\oint d\theta d\zeta J \phi}{\oint d\theta d\zeta J} \tag{7}$$

Let $J(\psi) = \oint d\theta d\zeta J$, then the left-hand side of Eq. (5) becomes

$$\langle \nabla_\perp^2 \phi \rangle = J(\psi)^{-1} \left(\partial_\psi J(\psi) \langle g^{\psi\psi} \rangle \partial_\psi \langle \phi \rangle + \partial_\psi J(\psi) \langle g^{\psi\psi} \partial_\psi \delta\phi \rangle + \partial_\psi J(\psi) \langle g^{\psi\theta} \partial_\theta \delta\phi \rangle \right) \tag{8}$$

The ratio of the second to the first term is $\epsilon \delta\phi_{m=1} / \langle \phi \rangle \ll 1$, where ϵ is the inverse aspect ratio of the tokamak. The third term is even smaller than the second one by the factor $g^{\psi\theta}/g^{\psi\psi} \ll 1$, which is the measure of non-orthogonality of the magnetic coordinates. Thus the second and the third terms in Eq. (8) can be neglected.

Thus, the gyrokinetic Poisson equation for the zonal component of the electrostatic potential reads

$$\begin{aligned} \partial_\psi J(\psi) \langle g^{\psi\psi} \rangle \partial_\psi \langle \phi \rangle &= \left[\begin{array}{c} -J(\psi) \langle \frac{1}{\rho_i^2} \rangle \frac{T_i}{n_i Z_i^2} \\ + \frac{T_i}{n_i Z_i^2} \partial_\psi J(\psi) \langle g^{\psi\psi} \rangle \partial_\psi \end{array} \right] \left(Z_i \langle n_i \rangle - e \langle n_e \rangle \right) & (9) \\ &\quad + \frac{T_i}{n_i Z_i^2} J(\psi) \partial_\psi \langle g^{\psi\psi} \partial_\psi \left(Z_i \langle n_i \rangle - e \langle n_e \rangle \right). & (10) \end{aligned}$$

Neglecting the coupling between magnetic field and $n_{n=0,m\neq 0}$ harmonics

$$\partial_\psi J(\psi) \langle g^{\psi\psi} \rangle \partial_\psi \langle \phi \rangle = \left[\begin{array}{c} -J(\psi) \langle \frac{1}{\rho_i^2} \rangle \frac{T_i}{n_i Z_i^2} + \\ \frac{T_i}{n_i Z_i^2} \partial_\psi J(\psi) \langle g^{\psi\psi} \rangle \partial_\psi \end{array} \right] \left(Z_i \langle n_i \rangle - e \langle n_e \rangle \right). \tag{11}$$

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The metric tensor element $g^{\psi\psi} = g_{\psi\psi}^{-1}$ can be found from

$$g_{\psi\psi} = (\partial_\psi X)^2 + (\partial_\psi Z)^2$$

using the spline functions X and Z. The Eq. 11 can be further integrated as

$$\begin{aligned} \partial_\psi \langle \phi \rangle &= \frac{T_i}{n_i Z_i^2} \partial_\psi \left(e \langle n_e \rangle - Z_i \langle n_i \rangle \right) - \frac{1}{J(\psi) \langle g^{\psi\psi} \rangle} & (12) \\ &\quad \int_0^\psi d\psi \frac{T_i}{n_i Z_i^2} \left[\begin{array}{c} J(\psi) \langle \frac{1}{\rho_i^2} \rangle \left(Z_i \langle n_i \rangle - e \langle n_e \rangle \right) \\ + J(\psi) \langle g^{\psi\psi} \rangle \frac{d \ln (T_i/n_i)}{d\psi} \partial_\psi \left(Z_i \langle n_i \rangle - e \langle n_e \rangle \right) \end{array} \right] & (13) \end{aligned}$$