Calculating α from B_{mn} in Boozer Coordinates

N.M. Ferraro and L. Shi

October 20, 2016

Notes on Boozer Coordinates

The Boozer coordinates constructed from M3D-C1 output use the coordinates (χ, θ, ζ), where χ increases outward from the magnetic axis, ζ increases counter-clockwise about the $R = 0$ axis, and θ increases clockwise about the magnetic axis in the (R, Z) plane. The (axisymmetric) equilibrium field is represented as

$$\vec{B}_0 = \psi' \left[\nabla\chi \times \nabla\zeta + q\nabla\theta \times \nabla\chi\right] \tag{1}$$

where $\psi' = d\psi/d\chi$. Note that $\vec{B}_0 \cdot \nabla\theta = \psi'\nabla\chi \times \nabla\zeta \cdot \nabla\theta = -\psi'\mathcal{J}^{-1}$.

Defining the poloidal magnetic flux Ψ as the magnetic flux through the area enclosed by the magnetic surface in the θ direction (*i.e.* upward along the $R = 0$ axis), we find

$$\begin{aligned} \Psi(\chi) &= \int_0^\chi d\chi \int_0^{2\pi} d\zeta\, \mathcal{J}\vec{B}_0 \cdot \nabla\theta \\ &= -2\pi\psi \end{aligned}$$

In covariant form, we may write the field as

$$\vec{B}_0 = \beta\nabla\chi + g(\chi)\nabla\zeta + I(\chi)\nabla\theta \tag{2}$$

The total toroidal current enclosed by a magnetic surface in the equilibrium is

$$\begin{aligned} I_\zeta(\chi) &= \frac{1}{\mu_0}\int d\vec{A} \cdot \nabla \times \vec{B}_0 \\ &= \frac{1}{\mu_0}\int_0^{2\pi} d\theta \int_0^\chi d\chi\, \mathcal{J}\nabla\zeta \cdot (\nabla\theta \times \nabla\chi)\left[\partial_\theta\beta - \partial_\chi I(\chi)\right] \\ &= \frac{2\pi}{\mu_0} I(\chi) \end{aligned}$$

Thus the covariant ζ component of the magnetic field $I(\chi)$ is related to the current enclosed by the surface $I_\zeta(\chi)$ by

$$I(\chi) = \frac{\mu_0}{2\pi} I_\zeta(\chi) \tag{3}$$

Similarly, the covariant θ component of the magnetic field $g(\chi)$ is related to the poloidal current external to the surface $I_\theta(\chi)$ by

$$g(\chi) = -\frac{\mu_0}{2\pi} I_\theta(\chi) \tag{4}$$

Note we may also calculate $I_\theta(\chi)$ in cylindrical (R, φ, Z) coordinates given the field representation $\vec{B} = \nabla\varphi \times \nabla\psi + F\nabla\varphi$ by

$$\begin{aligned} I_\theta(\chi) &= -\frac{1}{\mu_0}\int_0^{2\pi} d\varphi \int_0^R R'\, dR'\, \partial_{R'} F \\ &= -\frac{2\pi}{\mu_0} F \end{aligned}$$

Thus $g(\chi) = F(\chi)$.

Defining α to preserve $\delta\vec{B} \cdot \nabla\chi$

Many electromagnetic codes use a reduced model of electromagnetic perturbations in which

$$\begin{aligned} \delta\vec{B}_\alpha &= \nabla \times (\alpha \vec{B}_0) && (5) \\ &= \nabla\alpha \times \vec{B}_0 + \alpha\mu_0 \vec{J}_0 && (6) \end{aligned}$$

where we use the subscript α to denote the fact that this is a reduced representation.

In general, given $\delta\vec{B}$, there is no unique way to choose α, since $\delta\vec{B}$ has two degrees of freedom (three vector components plus the constraint that $\nabla \cdot \delta\vec{B} = 0$, whereas equation (5) has only one degree of freedom. Therefore we must choose α so that $\delta\vec{B}_\alpha$ captures the particular features of $\delta\vec{B}$ that we deem most important.

Here we choose to determine α such that $\delta\vec{B}_\alpha \cdot \nabla\chi = \delta\vec{B} \cdot \nabla\chi$. This choice ensures that the reduced field accurately represents B_{mn}, where

$$B_{mn} = \frac{(2\pi)^2}{A} \iint d\zeta\, d\theta\, \mathcal{J}\, (\delta\vec{B} \cdot \nabla\chi) e^{im\theta - in\zeta} \tag{7}$$

and where A is the surface area of the magnetic surface. Below, we show that α is uniquely determined by B_{mn}.

Writing the equilibrium field $\vec{B}_0$ in the covariant form defined in equation (2), we may write the reduced perturbed field as

$$\begin{aligned} \delta\vec{B}_\alpha &= \nabla\alpha \times \vec{B}_0 + \alpha\mu_0\vec{J}_0 \\ &= g(\chi)\left(\partial_\chi\alpha\nabla\chi \times \nabla\zeta + \partial_\theta\alpha\nabla\theta \times \nabla\zeta\right) \\ &+ I(\chi)\left(\partial_\chi\alpha\nabla\chi \times \nabla\theta + \partial_\zeta\alpha\nabla\zeta \times \nabla\theta\right) \\ &+ \beta\left(\partial_\theta\alpha\nabla\theta \times \nabla\chi + \partial_\zeta\alpha\nabla\zeta \times \nabla\chi\right) \\ &+ \alpha\mu_0\vec{J}_0 \end{aligned} \tag{8}$$

Noting that $\vec{J}_0 \cdot \nabla\chi = 0$, this becomes

$$\delta\vec{B}_\alpha \cdot \nabla\chi = \mathcal{J}^{-1}\left[g(\chi)\partial_\theta\alpha - I(\chi)\partial_\zeta\alpha\right] \tag{9}$$

Now we write α in terms of its spectral components

$$\alpha = \sum_{m,n} \alpha_{mn} e^{in\zeta - im\theta} \tag{10}$$

Inserting this definition into equation (9) yields

$$\delta\vec{B}_\alpha \cdot \nabla\chi = -i\mathcal{J}^{-1}\sum_{mn}\left[mg(\chi) + nI(\chi)\right]\alpha_{mn}e^{in\zeta - im\theta} \tag{11}$$

We now derive the relationship between α_{mn} and B_{mn} that results from the choice $\vec{B}_\alpha \cdot \nabla\chi = \vec{B} \cdot \nabla\chi$. Substituting the above expression for $\vec{B}_\alpha$ for $\vec{B} \cdot \nabla\chi$ in equation (7) yields

$$\begin{aligned} B_{mn} &= -\frac{i(2\pi)^2}{A}\iint d\zeta d\theta \sum_{m'n'}\left[m'g(\chi) + n'I(\chi)\right]\alpha_{m'n'}e^{i(m-m')\theta - i(n-n')\zeta} \\ &= -\frac{i(2\pi)^4}{A}\left[mg(\chi) + nI(\chi)\right]\alpha_{mn} \end{aligned} \tag{12}$$

Thus we find

$$\alpha_{mn} = \frac{iAB_{mn}}{(2\pi)^4\left[mg(\chi) + nI(\chi)\right]} \tag{13}$$

Other ways to define α

Defining α to preserve $\delta B_\parallel$

As mentioned in the previous section, the above definition of α is not unique since any choice of α will result in some loss of information about δB in general. Instead of defining α to retain exact information about $\delta\vec{B} \cdot \nabla\chi$, we may instead define α in order to exactly reproduce $\delta B_\parallel$. From the parallel component of equation (5) we find

$$\alpha = \frac{\delta\vec{B} \cdot \vec{B}_0}{\mu_0\vec{J}_0 \cdot \vec{B}_0} \tag{14}$$

Given this choice of α, the perpendicular components of a given $\delta\vec{B}$ will generally not agree with equation (5).

Defining α to preserve $\delta A_\parallel$

Since $\delta\vec{B} = \nabla \times \delta\vec{A}$, we find the relation between α and $\delta\vec{A}$:

$$\delta\vec{A} = \alpha\vec{B}_0 + \nabla\Phi \tag{15}$$

for some function Φ. Thus

$$\alpha = \frac{(\delta\vec{A} - \nabla\Phi)\cdot\vec{B}_0}{|B_0|^2} \tag{16}$$

If we make the assumption that $\vec{B}_0 \cdot \nabla\Phi = 0$ so that $\alpha = \delta A_{\|}/|B_0|$, we preserve the parallel component of the vector potential exactly, but all of the field components from equation (5) will generally differ from $\nabla \times \delta\vec{A}$.

Conversion from M3D-C1 Boozer coordinates to GTC Boozer coordinates

In GTC [1], the Boozer coordinatesa $(\bar{\psi}, \bar{\theta}, \bar{\zeta})$ are defined such that

$$\begin{aligned}\vec{B} &= \bar{\delta}\nabla\bar{\psi} + \bar{I}\nabla\bar{\theta} + \bar{g}\nabla\bar{\zeta} \\ &= q\nabla\bar{\psi}\times\nabla\bar{\theta} - \nabla\bar{\psi}\times\nabla\bar{\zeta}\end{aligned}$$

where $\bar{\psi}$ always increases outward from the magnetic axis, $\bar{\theta}$ is clockwise about the magnetic axis in positive-I_P equilibria anc counter-clockwise in negative-I_P equilibria (so that $\vec{B}\cdot\nabla\bar{\theta} = \bar{\mathcal{J}}^{-1} > 0$ is always satisfied), and $\bar{\zeta}$ always increases in the direction of I_P. This implies

$$\bar{\psi} = -\mathrm{sgn}(I_P)\,\chi\psi' \qquad \bar{\theta} = \mathrm{sgn}(I_P)\,\theta \qquad \bar{\zeta} = \mathrm{sgn}(I_P)\,\zeta \tag{17}$$

Thus

$$\bar{\delta} = -\mathrm{sgn}(I_P)\,\beta \qquad \bar{I} = \mathrm{sgn}(I_P)\,I \qquad \bar{g} = \mathrm{sgn}(I_P)\,g \tag{18}$$

For both codes, the safety factor q is defined such that $q > 0$ when I_P and B_T both have the same sign, and $q < 0$ otherwise.

GTC defines the Fourier components of α by

$$\alpha = \bar{\alpha}_{\bar{m}\bar{n}} e^{i\bar{m}\bar{\theta} - i\bar{n}\bar{\zeta}} \tag{19}$$

Using the transformation in equation (17) together with M3D-C1's definition of α_{mn} in equation (10) we find that this implies

$$\bar{m} = -\mathrm{sgn}(I_P)\,m \qquad \bar{n} = -\mathrm{sgn}(I_P)\,n \tag{20}$$

and therefore

$$\bar{\alpha}_{\bar{m}\bar{n}} = \begin{cases}\alpha^*_{mn} & I_P > 0 \\ \alpha_{mn} & I_P < 0\end{cases} \tag{21}$$

where we have used the fact that $\alpha_{-m\,-n} = \alpha^*_{mn}$

References

[1] P. Jiang, Z. Lin, I Holod, and C. Xiao. *Phys. Plasmas* **21**, 122513 (2014)