# Geometrical Correction to the Old Poisson Solver

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## 1 Introduction

In Dr. Lin's 1995 paper, a method of solving the gyrokinetic Poisson equation is developed. The calculation of the complicated average gyro-average potential, $\tilde{\phi}$, can be simplified to calculating the 4-point average of $\bar{\phi}$ on several rings. To calculate $\bar{\phi}$, we need to do the 4-point average again, which finally gives us:

$$\tilde{\phi}_{i,j} = \frac{1}{16}(4\phi_{i,j} + 2\phi_{i\pm2,j} + 2\phi_{i,j\pm2} + \phi_{i\pm2,j\pm2}) \tag{1}$$

Fig.1 (a) shows the eight extra points that appear in the equation above.

When we calculate $\tilde{\phi}$ in GTC, we neglect the redundant points in Fig.1 (a) and rotate the view by $\pi/4$. However, that's not the only difference. In Fig.1(b), in order to calculate $\tilde{\phi}$ in the middle, we need to calculate the values of $\phi$ at the eight red points outside. However, the coordinates in GTC is $(\psi, \theta)$ (the green lines in Fig.1(b)), but not Cartesian coordinate (the blue lines in Fig.1(b)), which means that we are actually calculating $\phi$ at the eight dark red points. Given $\phi$, GTC's calculation tends to give a systematic shift of $\tilde{\phi}$ inward, though they are the same to the lowest order.

Here I will derive the second order corrections in $(\psi, \theta)$ to put the eight points back to the correct positions. We do the coordinate transformation from Cartesian coordinate to polar coordinate (the black curves in Fig.1(b)) first, and then go from polar coordinate to flux coordinate.

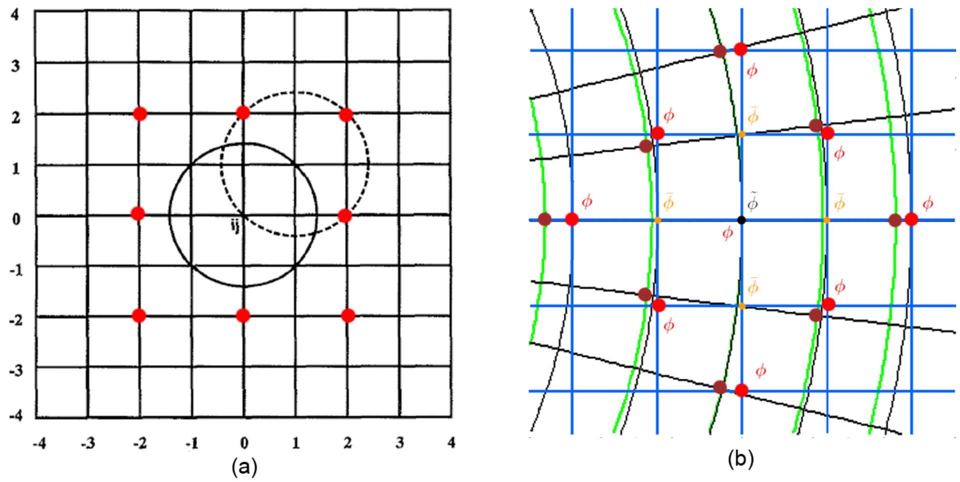


Figure 1: (a) The eight extra points we need to calculate $\tilde{\phi}_{i,j}$ (b) The difference between the points we want to use and the ones we actually use

# 2 Coordinate Transformation

Let me paraphrase the problem in this way:

*Without losing generosity, we can assume the original point $A$ is $(x_0, 0)$ in Cartesian coordinate, and $(\psi_0, 0)$ in flux coordinate. We know the coordinate of a point $A'$ near $A$ is $(x_0 + \Delta x, \Delta y)$. We want to express its coordinate in flux coordinate $(\psi_0 + \Delta\psi, \Delta\theta)$*

The solution of $(\Delta\psi, \Delta\theta)$ in the lowest order is trivial, and already calculated in GTC (`pgyro, tgyro`). So, in the end, I will express the solution in terms of their lowest order solution $(\Delta\psi_{(1)}, \Delta\theta_{(1)})$.

## 2.1 Cartesian coordinate to polar coordinate

$A'$ is $(x_0 + \Delta r, \Delta\theta)$ in polar coordinate. Let's expand the solution order by order.

$$\begin{cases} \Delta r = \Delta r_{(1)} + \Delta r_{(2)} + ... \\ \Delta\theta = \Delta\theta_{(1)} + \Delta\theta_{(2)} + ... \end{cases} \tag{2}$$

The transformation can be expressed as:

$$\begin{cases} (x_0 + \Delta r)^2 = (x_0 + \Delta x)^2 + (\Delta y)^2 \\ \Delta\theta = \text{arcTan}\left(\dfrac{\Delta y}{x_0 + \Delta x}\right) \end{cases} \tag{3}$$

The lowest order is trivial.

$$\begin{cases} \Delta r_{(1)} = \Delta x \\ \Delta\theta_{(1)} = \dfrac{\Delta y}{x_0} \end{cases} \tag{4}$$

The second order solution is

$$\begin{cases} \Delta r_{(2)} = \dfrac{\Delta y^2}{2x_0} = \dfrac{x_0}{2}\Delta\theta_{(1)}^2 \\ \Delta\theta_{(2)} = -\dfrac{\Delta x \Delta y}{x_0} = -\Delta\theta_{(1)}\Delta r_{(1)} \end{cases} \tag{5}$$

So the expression for $(\Delta r, \Delta\theta)$ is

$$\begin{cases} \Delta r = \Delta r_{(1)} + \dfrac{x_0}{2}\Delta\theta_{(1)}^2 + ... \\ \Delta\theta = \Delta\theta_{(1)} - \Delta\theta_{(1)}\Delta r_{(1)} + ... \end{cases} \tag{6}$$

## 2.2 polar coordinate to flux coordinate

Though in general flux coordinate could be quite complicated, I will only solve for the simplest one in cylindrical geometry with uniform $B$ field. $\theta$ stays the same as in polar coordinate.

$$\psi_0 + \Delta\psi = \frac{1}{2}B(x_0 + \Delta r)^2 \tag{7}$$

We can find that the lowest order is simply $\Delta\psi_{(1)} = Bx_0\Delta r_{(1)}$, and the second order is

$$\Delta\psi_{(2)} = B(x_0\Delta r_{(2)} + \frac{1}{2}\Delta r_{(1)}^2) = \frac{1}{2}Bx_0^2\Delta\theta_{(1)}^2 + \frac{1}{2}\frac{\Delta\psi_{(1)}^2}{Bx_0} \tag{8}$$

Now we rewrite $\Delta\theta_{(2)}$ in terms of ($\Delta\psi_{(1)}$ and $\Delta\theta_{(1)}$) :

$$\Delta\theta_{(2)} = -\Delta\theta_{(1)}\Delta r_{(1)} = -\frac{\Delta\theta_{(1)}\Delta\psi_{(1)}}{Bx_0^2} \tag{9}$$

So the coordinate of $A'$ is

$$\begin{cases} \Delta\psi = \Delta\psi_{(1)} + \dfrac{1}{2}Bx_0^2\Delta\theta_{(1)}^2 + \dfrac{1}{2}\dfrac{\Delta\psi_{(1)}^2}{Bx_0^2} + ... \\ \Delta\theta = \Delta\theta_{(1)} - \dfrac{\Delta\theta_{(1)}\Delta\psi_{(1)}}{Bx_0} + ... \end{cases} \tag{10}$$

## 3 Corrections

In this section, I will calculate the corrections of each point.

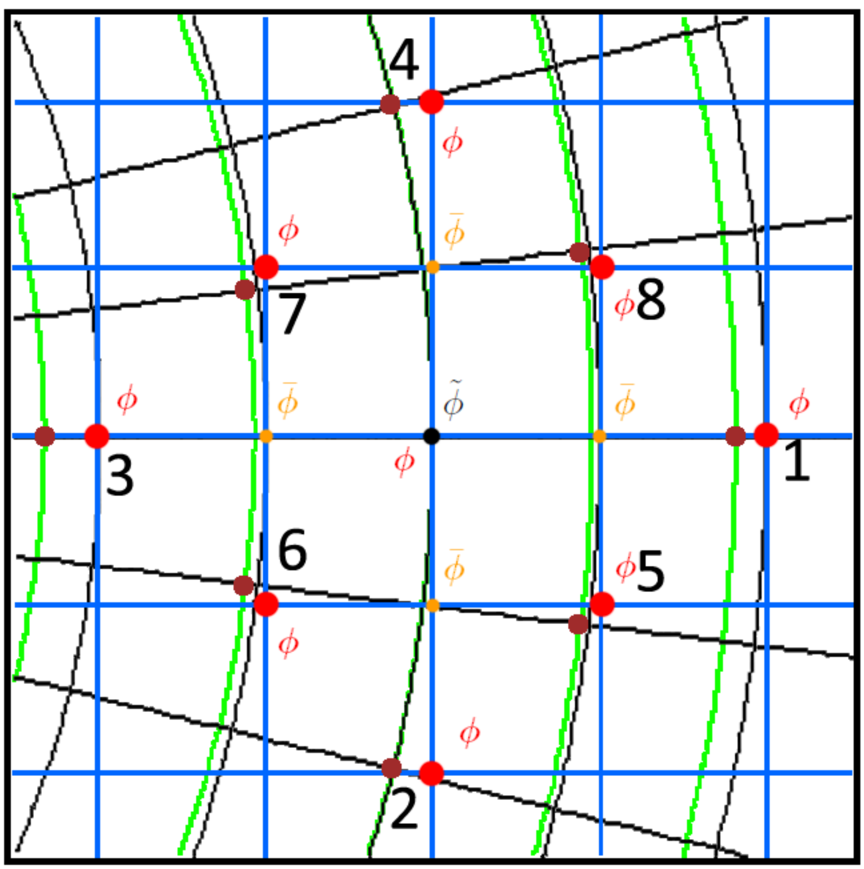


Figure 2: The corrections we need for the 8 points

The radius of the ring is $v\rho_s$. ($v$ is corresponding to `vring(kr)` in GTC)
We define

$$\Delta\Theta = \frac{1}{Bx_0^2}\Delta\Psi = v\rho_s \tag{11}$$

Mark the eight points with `kp` $= 1, 2, 3, ..., 8$, just like in GTC. (Fig.2)
For `kp` $= 1, 3$:

$$\begin{cases} \Delta\psi_{(1)} = \pm 2\Delta\Psi \\ \Delta\theta_{(1)} = 0 \end{cases} \Longrightarrow \begin{cases} \Delta\psi = \pm 2\Delta\Psi + 2\Delta\Theta\Delta\Psi + ... \\ \Delta\theta = 0 + ... \end{cases} \tag{12}$$

For `kp` $= 2, 4$:

$$\begin{cases} \Delta\psi_{(1)} = 0 \\ \Delta\theta_{(1)} = \pm 2\Delta\Theta \end{cases} \Longrightarrow \begin{cases} \Delta\psi = 2\Delta\Theta\Delta\Psi + ... \\ \Delta\theta = \pm 2\Delta\Theta + ... \end{cases} \tag{13}$$

For `kp` $= 5, 6$:

$$\begin{cases} \Delta\psi_{(1)} = \pm\Delta\Psi \\ \Delta\theta_{(1)} = \Delta\Theta \end{cases} \Longrightarrow \begin{cases} \Delta\psi = \pm\Delta\Psi + (\frac{1}{2} + \frac{1}{2})\Delta\Theta\Delta\Psi + ... \\ \Delta\theta = \Delta\Theta \mp \Delta\Theta^2 + ... \end{cases} \tag{14}$$

For `kp` $= 7, 8$:

$$\begin{cases} \Delta\psi_{(1)} = \mp\Delta\Psi \\ \Delta\theta_{(1)} = -\Delta\Theta \end{cases} \Longrightarrow \begin{cases} \Delta\psi = \mp\Delta\Psi + (\frac{1}{2} + \frac{1}{2})\Delta\Theta\Delta\Psi + ... \\ \Delta\theta = -\Delta\Theta \pm \Delta\Theta^2 ... \end{cases} \tag{15}$$

# 4 Check in simulation

## 4.1 Check the corrected coordinates by transforming back in $x, z$

In this part, I pick a particular point in GTC to check the eight gyro-average-needed points GTC gives us, and calculating the coordinate transformation by function `spx` and `spz` in GTC.

The central point is at (6.86E-02, 0.00).

uncorrected result

| kp | $x$ | $z$ |
|---|---|---|
| 1 | 7.55E-02 | 0.00 |
| 2 | 6.82E-02 | 7.10E-03 |
| 3 | 6.10E-02 | 0.00 |
| 4 | 6.82E-02 | -7.16E-03 |
| 5 | 7.20E-02 | 3.75E-03 |
| 6 | 6.48E-02 | 3.37E-03 |
| 7 | 6.48E-02 | -3.38E-03 |
| 8 | 7.20E-02 | -3.75E-03 |

corrected result

| kp | $x$ | $z$ |
|---|---|---|
| 1 | 7.58E-02 | 0.00 |
| 2 | 6.86E-02 | 7.14E-03 |
| 3 | 6.14E-02 | 0.00 |
| 4 | 6.86E-02 | -7.19E-03 |
| 5 | 7.22E-02 | 3.55E-03 |
| 6 | 6.50E-02 | 3.55E-03 |
| 7 | 6.50E-02 | -3.57E-03 |
| 8 | 7.22E-02 | -3.57E-03 |

There are many improvements here. For example, the $x$ of point 2 and point 4 should be the same as the $x$ of the central point. The $y$ of point 5 and point 6 should be the same. etc.

It's easy to check that the corrected result agrees with our expectation.

## 4.2 Check by Laplacian

GTC use $(\phi - \tilde{\phi})$ to calculate the Laplacian operation on $\phi$, $\nabla_\perp^2 \phi$. We know Bessel functions are eigenfunctions of Laplacian operator in cylindrical geometry. So given a $\phi$ in the form of a Bessel function, we know that $\nabla_\perp^2 \phi$ should have the same shape as $\phi$.

The result is shown below (Fig.3)

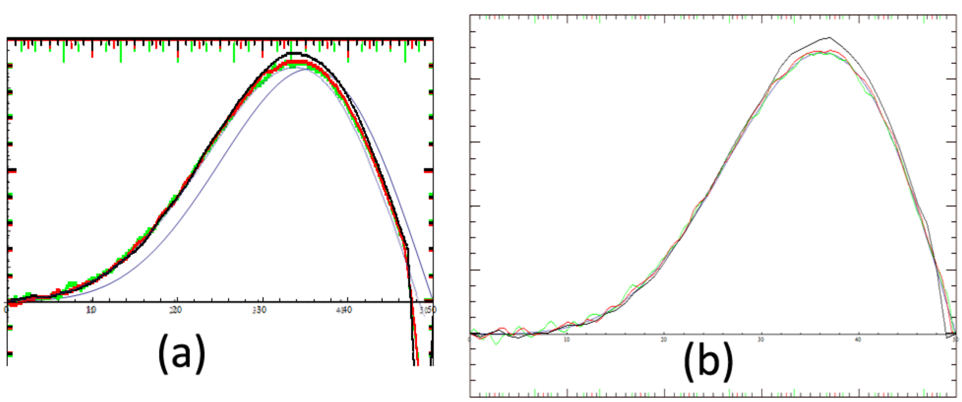


Figure 3: Comparison of the results from old and corrected Laplacian

Fig.3 (b) shows the simulation result from the corrected $\tilde{\phi}$ and (a) shows the simulation result from the corrected $\tilde{\phi}$.[1] In both figures, the blue curve represent the theoretical value. The black curve is the simulation result with 50 radial grid points. The red curve is the simulation result with 100 radial grid points. And the green curve is the simulation result with 150 radial grid points.

These result tells us that the deviation from theoretical curve we found in (a) is not a convergence issue but a systematic error in our simulation. And the corrected result agrees with theory very well.

However, we can see in both figures that when we increase the number of grid points, the noise in our result of Laplacian becomes larger and larger. But this is beyond the topic of this document. I will write about it if I fix it some day.

# 5 End

The main purpose of this document is to explain the complicated changes in the subroutine `poisson_solver_initial` and `gyroinitial` of GTC.

By now, these corrections are only validated in cylindrical geometry. It might be O.K. to apply these corrections to the toroidal geometry if we ignore the non-orthogonality between $\psi$ and $\theta$. But if this fails in toroidal geometry, this document provides a guideline of how to introduce some new geometric corrections.

[1] In (b), there is actually another change in the Laplacian near the edge which removes the singularity there in the original solver