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Does the orbit-averaged theory require a scale separation between periodic orbit size and perturbation correlation length?

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Using the canonical perturbation theory, we show that the orbit-averaged theory only requires a time-scale separation between equilibrium and perturbed motions and verifies the widely accepted notion that orbit averaging effects greatly reduce the microturbulent transport of energetic particles in a tokamak. Therefore, a recent claim [Hauff and Jenko, Phys. Rev. Lett. **102**, 075004 (2009); Jenko *et al.*, ibid. **107**, 239502 (2011)] stating that the orbit-averaged theory requires a scale separation between equilibrium orbit size and perturbation correlation length is erroneous. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4820804]

I. INTRODUCTION

The method of averaging[1] in classical mechanics strictly follows from the existence of an action integral (i.e., adiabatic invariant) in a quasi-periodic Hamiltonian system. This method was rigorously derived half a century ago using the canonical perturbation theory[2–4] by assuming a time-scale separation between equilibrium (periodic) and perturbed motions. The orbit-averaged theory for the guiding center (GC) drift motion has been widely applied in plasma physics,[5–9] and it predicts,[10–13] in particular, that energetic particle transport by microturbulence[14,15] in a tokamak is greatly reduced by the orbit averaging effects.

In a series of papers published recently in *Phys. Rev. Lett.*[16,17] and *Phys. Plasmas,*[18–20] Hauff, Jenko, and collaborators claimed that the orbit-averaged theory requires the orbit size (Δr) of the equilibrium GC drift in a tokamak to be smaller than the turbulence eddy size (λ_c). However, this claim was not supported by any mathematical proof. If this claim was correct, the requirement of a scale separation between equilibrium orbit size and perturbation correlation length (i.e., $\Delta r < \lambda_c$) in the orbit-averaged theory is of both fundamental and practical significance. Fundamentally, the requirement of the spatial-scale separation contradicts the textbook notion[1–13] that only a time-scale separation between equilibrium and perturbed motions is required by the orbit-averaged theory. Practically, the requirement of the spatial-scale separation leads to an energy scaling[16–20] different from that of the orbit-averaged theory[5,21–23] for the turbulent transport of energetic particles in burning plasmas such as ITER.

The radial orbit size (Δr) of the equilibrium GC drift in an axisymmetric tokamak was first defined in Eqs. (1)–(2) of Ref. 16

$$\Delta r \sim q\rho$$

for a passing particle, and

$$\Delta r \sim \frac{q\rho}{\sqrt{r/R_0}}$$

for a trapped particle (for simplicity, numerical coefficients of the order of unity are omitted). Here, q denotes the safety factor; ρ, the gyroradius; r, the minor radius; and R_0, the on-axis major radius. Ref. 16 defines the drift orbit size as "Δr *the diameter of the deviation from the flux surface in the radial r (or x) direction.*" Then, a condition for the orbit averaging (Eq. (3) of Ref. 16) was given as follows:

$$\frac{\max\{\mathrm{V_E}, |\mathrm{v_{dr}} - \mathrm{v_y}|\}\mathrm{T_{orbit}}}{\lambda_c} < 1 \text{ and } \mathrm{T_{orbit}} \ll \tau_c.$$

However, this condition was proposed without mathematical proof. This equation states that the orbital time of the equilibrium GC drift (T_{orbit}) should be shorter than the time taken by the GC to move across a turbulence correlation length (λ_c) at the maximal speed of the equilibrium drifts v_{dr} minus v_y, and perturbed E × B drift V_E. Ref. 16 defines v_{dr} as "*diamagnetic drifts with a velocity of the order of* $\rho_i c_i/R_0$," v_y as "*particle procession drift in the toroidal y direction,*" and c_i as "*ion thermal speed.*" Finally, considering that the equilibrium GC drifts are faster than the perturbed E × B drift for an energetic particle, Ref. 16 states that "*(for* $\Delta r \gtrsim \lambda_c$*) a particle gets decorrelated during its orbit motion, since it does not return into the correlated zone.*" An orbit decorrelation time now replaces the turbulence correlation time τ_c as the characteristic time scale for the wave-particle interaction. This orbit decorrelation time is given by

$$\tau^{\text{orbit}} = \frac{T_{\text{orbit}}\lambda_c}{\pi\Delta r}.$$

This equation (Eq. (4) of Ref. 16) clearly states that the time-scale separation requirement ($T_{\text{orbit}} < \tau^{\text{orbit}}$) directly leads to the spatial-scale separation requirement ($\Delta r < \lambda_c$). Ref. 16 concludes that "*orbit averaging becomes invalid almost as soon as the particle energies clearly exceed the thermal energy of the background plasma.*"

The above textual quotes and Eqs. (1)–(4) of Ref. 16 unambiguously claim that the orbit-averaged theory requires a spatial-scale separation ($\Delta r < \lambda_c$). Invoking this requirement, the authors of Refs. 16–20 then criticized the orbit-averaged theory in Ref. 21. The criticism[16–23] is primarily related to the application of the theory to the single particle dynamics, for which a first-principles theory exists.[1–4] Interestingly, in response to our comment[23] stating that the claim of a spatial-scale separation contradicts the canonical perturbation theory, Jenko *et al.*[17] vigorously defended the condition of the orbit-averaged theory as plainly expressed in Eqs. (1)–(4) of Ref. 16, but categorically denied "*a claim (namely, that orbit-averaged theory requires the smallness of the orbit size with respect to the turbulence correlation length), which was never made*."

The condition of the orbit-averaged theory expressed in Eq. (4) of Ref. 16 is clearly erroneous. As a very simple example, consider a time-independent, small amplitude electrostatic potential ϕ(r), which has a radial scale length that is smaller than the gyroradius of energetic particles. According to Eq. (4) of Ref. 16, the orbit-averaged theory is invalid when considering the effect of ϕ(r) on the guiding center motion in a tokamak. However, this contradicts the fact that the guiding center orbit is closed and periodic because of three constants of motion in this axisymmetric system: magnetic moment μ, energy E, and canonical toroidal angular momentum P_ζ.

The error in the claim made by the authors of Refs. 16–20 arises from the use of the equilibrium drift to calculate the "*orbit decorrelation time*" $\tau^{\rm orbit}$. In fact, in order to derive Eq. (4) from Eq. (3) of Ref. 16, v_{dr} or v_y must be the GC equilibrium drift due to the magnetic field gradient and curvature. Since the equilibrium drift determines the GC drift orbit size Δr, this procedure directly leads to the erroneous condition of a spatial-scale separation for the orbit-averaged theory. The spatial-scale separation claim is logically self-defeating: the effect of the orbit averaging is negligible, if $\Delta r < \lambda_c$. It also leads to some strange arguments regarding the GC dynamics. For example, Ref. 17 states that the applicability of the orbit-averaged theory could depend on whether "*the local safety factor happens to be an integer*."

In the one-dimensional canonical perturbation theory for a quasi-periodic Hamiltonian system,[1–4] the orbit-averaged theory for the GC drift motion is conceptually and mathematically equivalent to the gyro-averaged theory[24–28] (i.e., gyrokinetic theory) for cyclotron motion. If the heuristic arguments of Refs. 16–20 are applied to cyclotron motion, the orbital time of the equilibrium cyclotron motion should be shorter than an "*orbit decorrelation time*" defined as the time taken by a charged particle in moving across a turbulence eddy caused by the equilibrium cyclotron motion; the gyrokinetic theory would then be invalid for small turbulence eddy sizes regardless of the perturbation amplitude and frequency. This contradicts the standard gyrokinetic theory. In fact, the GC theory has been constructed[29] without any requirement of spatial-scale separation.

In this article, we use the canonical perturbation theory to show that orbit averaging is valid for an arbitrary GC drift orbit size in a tokamak. The orbit-averaged theory strictly follows from the existence of a longitudinal adiabatic invariant, which can only be broken by wave-particle resonances including linear and drift-bounce resonances and stochastic heating (see discussions of Eq. (18) in Sec. II). We demonstrate that the energy scaling of the turbulent transport of energetic particles predicted by the orbit-averaged theory is consistent with results from large scale gyrokinetic particle simulations. Our results verify the widely accepted notion that orbit averaging greatly reduces the microturbulent transport of energetic particles.

In Sec. II, canonical perturbation theory shows that the orbit-averaged theory does not require a spatial-scale separation. The energy scaling of the turbulent transport predicted by the orbit-averaged theory is shown to be fully consistent with results from large scale gyrokinetic particle simulations in Sec. III. Conclusions are drawn in Sec. IV.

II. CANONICAL PERTURBATION THEORY OF GC DYNAMICS

In this section, we show by using the canonical perturbation theory that the orbit-averaged theory does not require a spatial-scale separation. The Hamiltonian governing the GC drift in an axisymmetric tokamak is

$$H = H_0 + \delta H = \frac{1}{2}v_\parallel^2 + \mu B(r,\theta) + \phi(t,r,\theta,\zeta). \qquad (1)$$

Here, t denotes time and μ, magnetic moment. Particle mass and charge are taken to be unity. We consider a concentric tokamak for simplicity and use the toroidal coordinate system with the minor radius r, poloidal angle θ, and toroidal angle ζ. The equilibrium Hamiltonian $H_0 = \frac{1}{2}v_\parallel^2 + \mu B$ describes the periodic GC drift in the equilibrium magnetic field $B(r, \theta)$ and determines the GC orbit size Δr. The perturbed Hamiltonian $\delta H = \phi$ describes the perturbed GC drift due to the electrostatic potential $\phi(t, r, \theta, \zeta)$ with a correlation length λ_c. We neglect the finite Larmor radius effects and start with the GC Hamiltonian so that the magnetic moment is a constant even in the presence of the perturbation. The Hamilton's equation is constructed using the poloidal (P_θ) and toroidal (P_ζ) canonical angular momenta.[30]

The equilibrium GC drift orbit is determined by three constants of motion (μ, E, P_ζ) with energy $E = \frac{1}{2}v_\parallel^2 + \mu B(r, \theta)$. The Hamilton's equation can be greatly simplified through a canonical transformation from the $(\theta, P_\theta, \zeta, P_\zeta)$ phase space to action-angle variables. Obviously, μ is the first action associated with the cyclotron motion, which has been removed from the GC equation of motion (so that μ appears in the Hamiltonian only as a parameter). As per the standard procedure,[9,30,31] we define the second action (longitudinal invariant) of the GC transit or bounce motion as

$$J_2 = \frac{1}{2\pi}\oint P_\theta \mathrm{d}\theta = J_2(\mu, E, P_\zeta). \qquad (2)$$

The integration is performed along the complete path of a passing or trapped GC orbit. The corresponding angle θ_2 describes the GC position along the magnetic field line. The third action J_3 describes the toroidal precessional drift of the GC orbit center with (θ_3, J_3) representing the orbit center position in the toroidal and radial directions, respectively. In the action-angle phase space (θ_2, J_2, θ_3, J_3), the equilibrium Hamiltonian is simplified as

$$H_0 = E(\mu, J_2, J_3). \tag{3}$$

The transit or bounce frequency is

$$\omega_b = \frac{\partial H_0(\mu, J_2, J_3)}{\partial J_2}. \tag{4}$$

The precessional frequency is

$$\omega_p = \frac{\partial H_0(\mu, J_2, J_3)}{\partial J_3}. \tag{5}$$

The equilibrium GC drift is now completely described by the three actions (μ, J_2, J_3). A simple example[9] for this action-angle formulation is a purely passing particle with $\mu = 0$

$$H_0 = \omega_b J_2,\ J_2 = qR_0\sqrt{E/2},\ \text{and}\ \omega_b = \frac{\sqrt{2E}}{qR_0}. \tag{6}$$

On including the perturbation, the GC drift is no longer periodic and (J_2, J_3) are no longer constants of motion. We now seek to define the condition for the motion to remain quasi-periodic so that the canonical perturbation theory[2–4] can be applied. We use a small parameter ε to represent the deviation from the periodic motion

$$\varepsilon \ll 1. \tag{7}$$

The frequencies associated with the three actions are well separated for an energetic particle confined in a tokamak, i.e.,

$$\omega_p \sim \varepsilon\,\omega_b \sim \varepsilon^2\Omega. \tag{8}$$

Here, Ω denotes the particle cyclotron frequency. We assume a low frequency perturbation:

$$\omega \sim \varepsilon\omega_b \quad \omega = \left|\frac{1}{\phi}\frac{\partial\phi}{\partial t}\right|. \tag{9}$$

We can formally label the ε ordering of the terms in the Hamiltonian as

$$H = H_0(\mu, J_2, \varepsilon J_3) + \varepsilon\phi(\varepsilon t, r, \theta, \zeta). \tag{10}$$

We seek a near-identity transformation from (θ_2, J_2, θ_3, J_3) to $(\bar\theta_2, \bar J_2, \bar\theta_3, \bar J_3)$ so that the new Hamiltonian $\bar H$ does not depend on the rapidly varying angle $\bar\theta_2$. The canonical transformation is given by a generating function

$$S = \theta_2\bar J_2 + \theta_3\bar J_3 + \varepsilon S_1(\theta_2, \bar J_2, \theta_3, \bar J_3, \varepsilon t). \tag{11}$$

The coordinate transformation is given by

$$J_2 = \frac{\partial S}{\partial\theta_2} = \bar J_2 + \varepsilon\frac{\partial S_1}{\partial\theta_2}, \quad J_3 = \frac{\partial S}{\partial\theta_3} = \bar J_3 + \varepsilon\frac{\partial S_1}{\partial\theta_3}. \tag{12}$$

The new Hamiltonian is

$$\bar H = H_0(\mu, J_2, \varepsilon J_3) + \varepsilon\phi + \frac{\partial S}{\partial t}$$
$$= H_0(\mu, \bar J_2, \bar J_3) + \varepsilon\left(\phi + \varepsilon\frac{\partial S_1}{\partial t} + \omega_b\frac{\partial S_1}{\partial\theta_2} + \varepsilon\omega_p\frac{\partial S_1}{\partial\theta_3}\right). \tag{13}$$

In order to eliminate the rapidly varying angle $\bar\theta_2$ from the new Hamiltonian, we separate the perturbed potential into two components: $\phi = \bar\phi + \phi_1$. The orbit-averaged part $\bar\phi$ is defined as the potential integrated along the path of a complete equilibrium GC transit or bounce motion

$$\bar\phi(\varepsilon t, \mu, J_2, \theta_3, J_3) = \omega_b\oint\phi(\varepsilon t, r, \theta, \zeta)\frac{qR}{v_\parallel}\mathrm{d}\theta. \tag{14}$$

The new Hamiltonian is then cast order by order

$$\bar H = H_0(\mu, \bar J_2, \varepsilon\bar J_3) + \varepsilon\bar\phi(\varepsilon t, \mu, \bar J_2, \bar\theta_3, \bar J_3)$$
$$+ \varepsilon\left(\omega_b\frac{\partial S_1}{\partial\theta_2} + \phi_1\right) + \varepsilon^2\left(\frac{\partial S_1}{\partial t} + \omega_p\frac{\partial S_1}{\partial\theta_3} + \cdots\right). \tag{15}$$

We solve the following equation for the first-order solution of S_1:

$$\omega_b\frac{\partial S_1}{\partial\theta_2} + \phi_1 = 0. \tag{16}$$

The new Hamiltonian is independent of $\bar\theta_2$ when expressed up to the first order of ε as

$$\bar H = H_0(\mu, \bar J_2, \varepsilon\bar J_3) + \varepsilon\bar\phi(\varepsilon t, \mu, \bar J_2, \bar\theta_3, \bar J_3). \tag{17}$$

The above is the desired Hamiltonian that describes the perturbed GC bounce or transit motion under the influence of the orbit-averaged potential $\bar\phi$. The new action $\bar J_2$ is an adiabatic invariant. As expected, the orbit-averaged theory strictly follows from the existence of this new longitudinal invariant.

We now list the assumptions made in the derivation leading to Eq. (17)

$$\omega,\ \omega_d,\ \omega_{nl} \sim \varepsilon\omega_b. \tag{18}$$

First, the condition $\omega \sim \varepsilon\omega_b$ results from the assumption that the $\frac{\partial S_1}{\partial t}$ term in Eq. (15) is of the second order. The violation of this condition leads to the linear resonance. Second, the drift frequency $\omega_d = n\omega_p$ is the Doppler-shifted frequency of the perturbed potential with a toroidal mode number n, as observed from the GC orbit center with a toroidal precession frequency of ω_p. The condition of $\omega_d \sim \varepsilon\omega_b$ results from the assumption that the term $\omega_p\frac{\partial S_1}{\partial\theta_3}$ in Eq. (15) is of the second order. The violation of this condition leads to the familiar drift-bounce resonance underlying the ripple loss process.[32] Finally, the nonlinear frequency $\omega_{nl} = \frac{\partial\bar\phi}{\partial\bar J_2}$ is associated with the perturbed GC drift, which depends on the turbulence amplitude. The condition of $\omega_{nl} \sim \varepsilon\omega_b$ results from the assumption that the perturbed potential ϕ is smaller than the equilibrium Hamiltonian. The violation of this condition requires a large turbulence intensity, which can lead to stochastic heating.[33]

These conditions (Eq. (18)) for the orbit-averaged theory impose only a time-scale separation requirement. Therefore, the canonical perturbation theory of the GC dynamics does not require the equilibrium orbit size to be smaller than the perturbation correlation length, in contradiction to the claim in Refs. 16–20.

III. ENERGY SCALING OF ENERGETIC PARTICLE TRANSPORT BY ELECTROSTATIC MICROTURBULENCE

Having established the validity of the orbit averaging, we now show that the energy scaling of the turbulent transport predicted by the orbit-averaged theory is fully consistent with results from large scale gyrokinetic particle simulations. Assuming a small perturbation amplitude, the gyrokinetic quasilinear theory with the orbit-averaging for electrostatic turbulence in an axisymmetric system has been rigorously derived[5] using the nonlinear gyrokinetic theory.[25] Here, we apply the theory of Ref. 5 to the diffusivity for purely passing particles ($\mu = 0$) in a tokamak to obtain

$$D_p \propto \frac{c^2}{B^2} \sum_{n,l} |k_\theta \phi_n J_l(k_\perp \rho_d)|^2 \;\; \delta(\omega - \bar{k}_\| v_\| - l\omega_b) \tag{19}$$

and for deeply trapped particles ($v_\| \sim 0$),

$$D_t \propto \frac{c^2}{B^2} \sum_{n,l} |k_\theta \phi_n J_0(k_\perp \rho_E) J_l(k_\perp \rho_d)|^2 \;\; \delta(\omega - n\omega_p - l\omega_b). \tag{20}$$

Here, ρ_E and ρ_d denote the gyroradius and GC drift orbit width, respectively; k_θ and $k_\perp$ denote the poloidal and perpendicular wavenumbers; c denotes the speed of light; l denotes the transit/bounce harmonics; and the over-bar indicates averaging over the GC transit/bounce orbit on the $1/\omega_b$ fast time scale. The parallel wavevector $\bar{k}_\| = (n - m/q)/R$ is evaluated at the transit orbit center (m is the poloidal harmonic). The Bessel function $J_0^2(k_\perp \rho_E)$ results from the gyroradius and $J_l^2(k_\perp \rho_d)$ from the width of the GC transit/bounce orbit. The only assumption made in deriving these equations is time-scale separation between the three degrees of freedom in the equilibrium motion (Eq. (8)), i.e., $\Omega \gg \omega_b \gg \omega_p$, together with the usual quasilinear assumptions of a small amplitude turbulence made in Refs. 16–23. Equations (19) and (20) describe the GC orbit-averaged theory with an additional assumption of a time-scale separation between equilibrium and perturbed motions, as expressed in Eq. (18).

Purely passing particles—For energetic particles, the $l = 0$ term in Eq. (19) dominates for a microturbulence with modestly ballooning structures.[34] In such a case, Eq. (19) becomes

$$D_p \propto \frac{c^2}{B^2} \sum_{n} |k_\theta \phi_n J_0(k_\perp \rho_d)|^2 \;\; \delta(\omega - \bar{k}_\| v_\|). \tag{21}$$

The GC orbit-averaged potential $\bar{\phi} = \phi_n J_0(k_\perp \rho_d)$ is therefore valid for arbitrary turbulence eddy sizes.

Deeply trapped particles—For low-n modes (large poloidal eddy size) with arbitrary radial eddy sizes, the dominant term in Eq. (20) is the $l = 0$ term. In such a case, Eq. (20) thus becomes

$$D_t \propto \frac{c^2}{B^2} \sum_{n} |k_\theta \phi_n J_0(k_\perp \rho_E) J_0(k_\perp \rho_d)|^2 \;\; \delta(\omega - n\omega_p), \tag{22}$$

which is simply the result of usual bounce-averaged theory as applied previously to the trapped-ion mode[34] and fishbone oscillation.[35] The GC orbit-averaged potential $\bar{\phi} = \phi_n J_0(k_\perp \rho_E) J_0(k_\perp \rho_d)$ is therefore valid for arbitrary radial eddy sizes.

For high-n modes, the dominant term in Eq. (20) is the $l = 1$ term for energy ranges of α-particles in ITER. This corresponds to the drift-bounce resonance $\omega - n\omega_p - \omega_b = 0$. The usual adiabatic invariant $\bar{J}_2$ is not conserved, in this case, due to the resonance between GC parallel and perpendicular equilibrium motions. Again, this is completely analogous to the breaking of μ in the cyclotron motion due to the resonance between the perpendicular gyro-frequency and parallel Doppler-shifted frequency ($\Omega = k_\| v_\|$). Using the secular perturbation theory for two degrees of freedom, Taylor and Laing[36] showed that a new adiabatic invariant can be constructed by removing the resonance through a transformation to the Doppler-shifted frame. This is always possible as long as the turbulence intensity is not strong enough for the onset of global stochasticity.[4] When ω, $\omega_{nl} \ll n\omega_p, \omega_b$, the same procedure can be used to remove the GC drift-bounce resonance through a transformation to a new frame rotating with a precessional frequency ω_p; in the new frame the GC orbit is closed after a complete bounce motion. Then, a modified longitudinal invariant can be constructed (superadiabaticity) and an orbit integration over the fast bounce motion gives a perturbed Hamiltonian with $\bar{\phi} = \phi_n J_0(k_\perp \rho_E) J_1(k_\perp \rho_d)$. Thus, the orbit integration leads to an effective orbit-averaged potential.

Energy scaling—We now demonstrate that the energy scaling for the energetic particle transport with orbit averaging (Eqs. (19)–(22)) is fully consistent with results of large scale gyrokinetic particle simulations of electrostatic ion temperature gradient (ITG) turbulence using global gyrokinetic toroidal code (GTC).[37] We note that the GTC simulation results[21,22] are the first published phase space structures of energetic particle diffusivity, and can be used to construct the diffusivity of any distribution function. For example, the diffusivity of a slowing down beam, in qualitative agreement with an independent study,[13] successfully explains many features of the fast ion transport in DIII-D tokamak plasmas.[14,15]

For purely passing energetic particles, Eq. (21) gives an asymptotic energy scaling of the diffusivity, $D_p \propto E^{-1}$, which agrees very well with the GTC simulation results in Fig. 1. To verify that the resonance condition of $\omega = \bar{k}_\| v_\|$ and the orbit averaging $J_0^2(k_\perp \rho_d)$ each contributes an energy dependence of $E^{-1/2}$ to the diffusivity, we examine the energy dependence of the Bessel function J_0^2 by numerically integrating the intensity weighted J_0^2 over k_θ (dominant contribution to $k_\perp$) using the turbulence spectrum measured from simulations. The result in Fig. 2 indicates that when $E > 8.2\,T$, J_0^2 exhibits $E^{-1/2}$ scaling. The other condition, $\omega_b > \omega$ is easily satisfied when $E > 1.2\,T$.

For deeply trapped energetic particles, the resonance condition becomes $n\omega_p = l\omega_b$ (dominated by $l = 1$), which contributes an energy scaling of E^{-1} to the diffusivity. The particle gyro-averaging $J_0^2(k_\perp \rho_E)$ and GC orbit averaging $J_1^2(k_\perp \rho_d)$ each contributes to a dependence of $E^{-1/2}$.

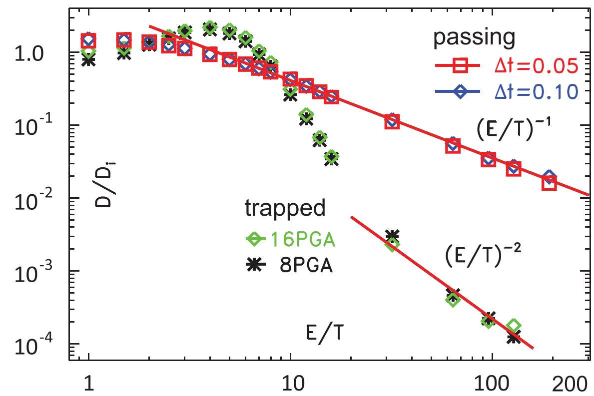

FIG. 1. Diffusivity D (normalized using the thermal ion D_i) as a function of particle energy E (normalized using plasma temperature T). Convergences of time step (Δt) for passing particles and gyro-averaging (8 and 16 points) for trapped particles are demonstrated.

Therefore, deeply trapped particle diffusivity scales as $D_t \propto E^{-2}$, which is confirmed by the GTC simulation results in Fig. 1. We now estimate the energy threshold for this asymptotic scaling. For the ITG turbulence, we find that when $E > 13\,T$, the condition $\omega_b > \omega$ is satisfied. Since $\rho_d \gg \rho_E$, J_1 reaches the asymptotic limit of the $E^{-1/2}$ dependence faster than J_0.

In contrast, the erroneous claim in Refs. 16–20 leads to an energy scaling of the diffusivity $D_t \propto E^{-1.5}$ for trapped particles. Moreover, GENE simulation results (Fig. 2 of Ref. 16) show that the diffusivity of trapped particles is larger than that of passing particles, i.e., $D_t > D_p$ at $E = 100\,T$ (roughly the energy of α-particles in ITER). This clearly contradicts the orbit-averaged theory and GTC simulation results in Fig. 1, which show that $D_t < 0.01 D_p$ at $E = 100\,T$. Therefore, the validity of GENE simulation results in Refs. 16–20 for predicting the scaling and the level of transport of energetic particles by microturbulence in ITER is questionable.

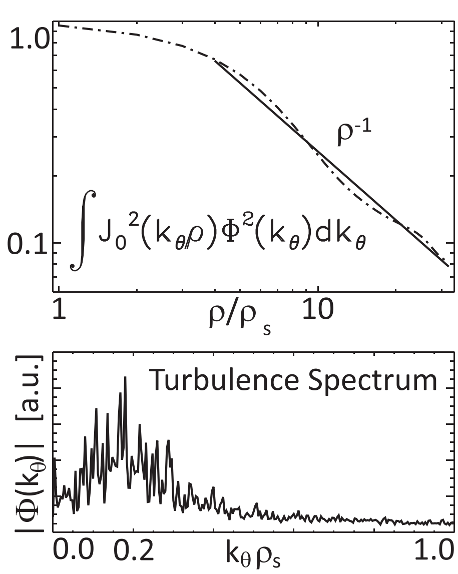

FIG. 2. Intensity-weighted Bessel function J_0^2 as a function of gyroradius ρ (dashed line in upper panel) calculated using the turbulence spectrum (lower panel) from simulations. A fit to $1/\rho$ is plotted as a solid line in the upper panel.

We remark that the accuracy in measuring D_t must be better than 10^{-4} times the thermal diffusivity D_i since $D_t \sim 10^{-4} D_i$ at $E = 100\,T$ as can be seen from Fig. 1. Such a level of accuracy is very difficult to achieve since most gyrokinetic simulations do not bother to reach an accuracy of even 10^{-2}. In fact, special techniques and cares[38–41] need to be adopted in the GTC simulation[21,22] to obtain an extremely high accuracy in measuring D_t; these techniques include the detailed Lagrangian analysis, and the convergences of particle number, time step, device size, and gyro-averaging, etc.

IV. CONCLUSIONS

In summary, we have shown using the canonical perturbation theory that a recent claim[16–20] stating that the orbit-averaged theory requires a scale separation between the equilibrium orbit size and perturbation correlation length is erroneous. The orbit-averaged theory only requires a time-scale separation. We have rigorously derived the orbit averaging for the purely passing and deeply trapped energetic particles in an electrostatic turbulence with an arbitrary eddy size. Our results verify the widely accepted notion that particle gyro-averaging and GC orbit-averaging greatly reduce the energetic particle transport by microturbulence.

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