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Nonlinear toroidal mode coupling: a new paradigm for drift wave turbulence in toroidal plasmas

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Abstract

Global gyrokinetic particle simulations and nonlinear gyrokinetic theory indicate that electron temperature gradient (ETG) instability saturates via nonlinear toroidal coupling. In such nonlinear interactions, the wave energy at the unstable high toroidal-mode number domain cascades towards the more stable lower toroidal-mode number domain via scatterings off the driven low-mode number quasi-modes. During the saturation process, there is little zonal flow generation and the radial fluctuation envelopes maintain extended structures. The nonlinear coupling process depends critically on the toroidal geometry and, as such, represents a new paradigm for the spectral cascade of drift wave turbulence in toroidal systems.

1. Introduction

Recent numerical simulations have demonstrated that nonlinear saturation of the electron temperature gradient (ETG) instability is due to nonlinear toroidal couplings [1, 2]. Unlike in the case of ion temperature gradient (ITG) modes [3–6], there is little zonal flow (ZF) generation [7] during the saturation process and the radial fluctuation envelopes maintain extended structures [1, 2], known as *streamers*. Simulation results also indicate that the dominant nonlinear interaction is due to couplings of fluctuations with different toroidal mode numbers, n. In fact, the saturation level of a single n mode is significantly higher than that of two ns. Furthermore, the nonlinear ETG fluctuation spectrum is characterized by a *nonlocal inverse cascade process*, where the longest wavelengths are generated first [1, 2].

Understanding these results and providing a theoretical framework to explain these peculiar ETG dynamics is the main motivation of the present work. In section 2, we derive the spectral transfer equations for ETG turbulence in a framework that is sufficiently general so that it may be readily extended to the corresponding dynamics of the ITG case. These equations demonstrate the existence and the truly *toroidal nature* of the nonlocal inverse spectral cascade process via drift wave turbulence scattering off the driven low-mode number quasi-modes, which have the role of *mediators* [1,2]. Despite its general character, this process dominates the nonlinear ETG dynamics since it occurs on a typically much shorter time scale than that of spontaneous ZF generation, as shown in section 3; however, the same process becomes subdominant for ITG dynamics with respect to ZF–ITG nonlinear interactions. This nonlinear ETG–ITG symmetry breaking [8,9] is due to the different ion and electron dynamic responses to ZFs (see section 3).

The qualitative features of the saturated ETG spectrum, as expected from the nonlinear toroidal coupling paradigm, are discussed in section 4, and concluding remarks are given in section 5.

2. Nonlinear toroidal coupling and ETG spectral transfer equations

Assume a right-handed toroidal flux coordinate system (r, ϑ, ζ) with straight field lines such that $(\boldsymbol{B} \cdot \nabla\zeta)/(\boldsymbol{B} \cdot \nabla\vartheta) = q(r)$. Consider also a long wavelength linear ETG unstable spectrum, characterized by high toroidal mode numbers, $n \gg 1$ and $k_\vartheta \rho_e = O(n^{-1/4})$, and by typical wave-vector spectral width $\Delta k_\vartheta / k_\vartheta = O(n^{-1/2})$. Here, k_ϑ stands for the ETG poloidal wave-vector, while ρ_e indicates the electron Larmor radius. Since the ETG frequency has an approximately linear dependence on the mode number, the typical frequency mismatch is $\Delta\omega/\omega = O(n^{-1/2})$. It has been shown in [1,2], with the present wavelength and frequency ordering and assuming proximity to marginal stability, that the normalized linear growth rate is $(\gamma/\omega) = O(k_\vartheta \rho_e) = O(n^{-1/4})$. The nonlinear electron gyrokinetic equation [10] can then be solved in the fluid limit and the ETG quasi-neutrality condition can be cast into the following form,

$$\partial_t L_k \delta\phi_k = \frac{c}{2B} \alpha_e \rho_e^2 \sum_{\boldsymbol{k}'+\boldsymbol{k}''=\boldsymbol{k}} \boldsymbol{b} \cdot \left(\boldsymbol{k}''_\perp \times \boldsymbol{k}'_\perp\right) \left(k''^2_\perp - k'^2_\perp\right) \delta\phi_{k'} \delta\phi_{k''}, \tag{1}$$

where the ion response is taken to be adiabatic [1,2], L_k is the linear ETG wave-operator, $\alpha_e = 1+\tau(1+\eta_e)/[(3\tau-1)L_n/R+1/2]$ [4], $\tau = T_e/T_i$, $\eta_e = (\partial \ln T_e/\partial \ln n_e)$, $L_n^{-1} = -\partial_r \ln n_e$ and R is the torus major radius. More precisely, L_k is given by

$$L_k = 1 + \tau + (T_e/en_e)\delta\phi_k^{-1} \langle J_0(k_\perp v_\perp/\omega_{ce}) \delta g_{ke}^{\text{linear}} \rangle, \tag{2}$$

where angular brackets indicate velocity space integration, J_0 is the Bessel function, ω_{ce} is the electron cyclotron frequency and $\delta g_{ke}^{\text{linear}}$ is the linearized nonadiabatic electron response to ETG fluctuations, having separated, as usual, the adiabatic part in the fluctuating electron distribution function:

$$\delta F_{ke} = (e\delta\phi_k/T_e) F_{0e} + \exp(-\mathrm{i}\boldsymbol{k}_\perp \cdot \boldsymbol{v} \times \boldsymbol{b}/\omega_{ce}) \delta g_{ke}. \tag{3}$$

Note that assuming $|\Delta\omega/\omega| \approx |\Delta k_\vartheta/k_\vartheta| \ll |\gamma/\omega|$ is equivalent to considering the spectrum of modes for which the linear growth rate is not significantly degraded from its maximum value, consistent with the initial value approach of nonlinear gyrokinetic codes.

The nonlinear toroidal coupling paradigm is based on the interaction of two different ETG toroidal modes and generating preferentially a low frequency, low toroidal mode number quasi-mode with $n_\ell = O(n^{1/2})$ [1,2]. The reason for this ordering is that the typical linear wave-vector spectral width is $\Delta k_\vartheta/k_\vartheta = O(n^{-1/2})$. The generation of the high frequency, high toroidal mode number wave is less efficient due to the higher intrinsic inertia [1,2]. Three-wave resonant coupling is also inefficient, since the ETG frequency has an approximately linear dependence on the mode number; thus, the frequency mismatch $\Delta\omega/\omega = \mathrm{O}(n^{-1/2})$ is much smaller than $(\gamma/\omega) = \mathrm{O}(k_\vartheta\rho_\mathrm{e}) = \mathrm{O}(n^{-1/4})$ and the low frequency, low toroidal mode number wave behaves as a quasi-mode [1,2]. In summary, the ETG fluctuations are composed of two *plasmon* distributions: one of the high-n ($n = \mathrm{O}(10^2)$) unstable or weakly damped ETG modes and one of the low-n_ℓ forced *meso-scale* quasi-modes. While $n_\ell = \mathrm{O}(n^{1/2})$ [1,2] gives the typical wavelength ordering of the quasi-mode spectrum, the maximum value of n_ℓ, i.e. $n_\ell \lesssim n_{\ell\mathrm{m}}$, is set by the forced nature of the quasi-modes. Thus, using $\omega \propto n$, $n_{\ell\mathrm{m}}$ is set by the condition $\gamma \gtrsim \omega_\ell \simeq (n_\ell/n)\omega$, yielding $n_{\ell\mathrm{m}} \simeq (\gamma/\omega)n = \mathrm{O}(n^{3/4})$. Note that numerical simulations show that the lower end of the linear ETG unstable spectrum is at $k_\vartheta\rho_\mathrm{e} \approx 10^{-1} = \mathrm{O}(n^{-1/2})$ [1,2], i.e. that ETG are marginally stable at $n \lesssim n_{\mathrm{MS}}$, with $n_{\mathrm{MS}} = \mathrm{O}(n^{3/4}) \approx n_{\ell\mathrm{m}}$. In section 4, we demonstrate that this point has important consequences for the shape of the saturated ETG spectrum. Efficient nonlinear mode coupling requires that the localized radial mode structures of two ETG modes, typical of toroidal drift waves, have significant overlap about the same rational surface. This requirement is equivalent to a *selection rule* for nonlinear toroidal coupling. In fact, considering two ETG modes with mode numbers (m_0, n_0) and $(m_1, n_1) = (m_0 - m_\ell, n_0 - n_\ell)$, they are characterized by significant radial overlap if and only if $m_0/n_0 = (m_0 - m_\ell)/(n_0 - n_\ell) + \mathrm{O}(n_0^{-1}n_\ell^{-1}) = m_\ell/n_\ell + \mathrm{O}(n_0^{-1})$, i.e. if the ETG modes are localized near a low order rational surface, r_0, where the safety factor $q(r_0) = q_0 = q_\ell + \mathrm{O}(n_0^{-1})$. When this condition is satisfied, all modes with $(m_0 + jm_\ell, n_0 + jn_\ell)$ and $j = 0, \pm1, \pm2, \ldots$ but $|j| \ll n_\ell = \mathrm{O}(n_0^{1/2})$ are characterized by significant radial overlap. Thus, the selection rule for efficient nonlinear toroidal mode coupling is equivalent to the *coarse graining* of the q-profile and imposes a lower bound for n_ℓ at $n_{\mathrm{c.o.}} = \mathrm{O}(n_0^{1/2})$, essentially set by the value of q_0. Furthermore, given n_ℓ, the quasi-mode can be conceived as the *incoherent* superposition of the resulting action of the various relevant ETG toroidal mode pairs, since the combined effect of, say, $(m_0, n_0) \times (m_0 - m_\ell, n_0 - n_\ell)^*$ and $(m_0 + 2m_\ell, n_0 + 2n_\ell) \times (m_0 + m_\ell, n_0 + n_\ell)^*$ tends to rapidly phase mix in the nonlinear stage. Meanwhile, given two ETG modes with toroidal number n_0 and n_1, characterized by significant overlap at $r = r_0$ due to the condition $m_0/n_0 = (m_0 - m_\ell)/(n_0 - n_\ell) + \mathrm{O}(n_0^{-1}n_\ell^{-1})$, all $\approx n_0^{1/2}$ poloidal harmonics of the toroidal modes will also overlap across the radial region where the modes are localized [11] (see equation (4)).

The ETG mode structures can be represented in the ballooning space as [12,13]

$$\begin{aligned}
\delta\phi_k &= \mathrm{e}^{-\mathrm{i}n\zeta} A_k(r,t) \sum_m \mathrm{e}^{\mathrm{i}m\vartheta}\Phi_k(nq - m),\\
\Phi_k(nq - m) &= \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}(nq-m)\theta}\hat{\Phi}_k(\theta)\mathrm{d}\theta,\\
\hat{\Phi}_k(\theta) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}(nq-m)\theta}\Phi_k(nq - m)\mathrm{d}(nq),
\end{aligned} \tag{4}$$

where the radial envelope $A_k(r,t)$ accounts for slow radial dependences on the scale $\approx n^{-1/2}L_n$, while the parallel mode structure $\hat{\Phi}_k(\theta)$ describes fast radial variations on the $\approx (nq')^{-1}$ scale, due to $k_\parallel$ changes in a sheared magnetic field. As stated in section 1, our analysis stems from the evidence of numerical simulation results [1,2] suggesting that ETG saturation is due to nonlinear toroidal mode coupling with a negligible effect on the radial wave envelope, consistent with the weak ZF generation rate via modulational instability [3,4,9] (see section 3). For this reason, we can neglect the $A_k(r,t)$ variation on the $\approx n^{-1/2}L_n$ scale and, similarly, we can assume

$$\delta\phi_\ell = \mathrm{e}^{-\mathrm{i}n_\ell\zeta+\mathrm{i}m_\ell\vartheta}A_\ell(t)\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}(n_0q-m_0)\theta}\hat{\Phi}_\ell(\theta)\mathrm{d}\theta, \tag{5}$$

having neglected variations on the $\approx (n_\ell q')^{-1}$ scale. Note that, with equation (5), we have emphasized the fast radial dependences of $\delta\phi_\ell$ on scales $\approx (n_0 q')^{-1}$ and shorter, which can be shown to dominate the nonlinear coupling, consistent with the nonlinear generation mechanism of the quasi-modes [1, 2, 14]. Such fine radial structures justify the adiabatic treatment of the ion response to low toroidal mode numbers, n_ℓ, and make it possible to use the representation of equation (5), which is based only on the radial scale separation between local fluctuations and their envelope. Meanwhile, they demonstrate that quasi-modes are characterized by filamentary (non-ballooning) structures along the magnetic field lines, with $|qRk_\parallel| = \mathrm{O}(n^{-1/2})$, i.e. with an extremely long parallel wavelength.

The n_ℓ quasi-mode can be thought of as the incoherent superposition of the nonlinear toroidal coupling of $(m_0 + jm_\ell, n_0 + jn_\ell) \times (m_0 + jm_\ell - m_\ell, n_0 + jn_\ell - n_\ell)^*$ ETG mode pairs, with $j = 0, \pm 1, \pm 2, \ldots$, or, in short, $(m_{0j}, n_{0j}) \times (m_{1j}, n_{1j})^*$. Thus, introducing the notation $a_k(t)\mathrm{e}^{-\mathrm{i}\omega_k t} = (eA_k/T_\mathrm{e})$ and $\hat{\alpha}_\mathrm{e} = \alpha_\mathrm{e}|\omega_\mathrm{ce}|\rho_\mathrm{e}^4$, the nonlinear equation for the n_ℓ quasi-mode, obtained from equation (1), becomes [1,2,9]

$$(-\mathrm{i}\omega_\ell + \partial_t)a_\ell(t)\hat{L}_\ell\hat{\Phi}_\ell(\theta) = 2\pi\hat{\alpha}_\mathrm{e}s\theta\sum_j\sum_l k_{\vartheta 0j}k_{\vartheta 1j}a_{0j}(t)a_{1j}^*(t)\delta(\theta - 2\pi l) \times \int_{-\infty}^{+\infty}(1 + s^2\eta^2)(k_{\vartheta 1j}^2\hat{\Phi}_{0j}(\eta_l)\hat{\Phi}_{1j}^*(\eta) - k_{\vartheta 0j}^2\hat{\Phi}_{0j}(\eta)\hat{\Phi}_{1j}^*(\eta_l))\mathrm{d}\eta, \tag{6}$$

with $\hat{L}_\ell$ the ballooning space representation of the linear ETG wave-operator [1,2], $s = r_0 q_0'/q_0$ the magnetic shear, $\eta_l = \eta - 2\pi l$ and the ETG parallel mode structures considered by us as characterized by a given parity for up–down symmetric equilibria. Furthermore, we have neglected variations on the $\approx (n_\ell q')^{-1}$ scale, consistent with previous discussions, and have used the fact that the ETG frequency has an approximately linear dependence on the mode number. Note that the l-summation in equation (6) accounts for the rapidly changing radial structure of the quasi-modes on scales shorter than $\approx (n_0 q')^{-1}$. In equation (6), we have assumed the following representation,

$$\begin{aligned}(e/T_\mathrm{e})\delta\phi_{0j} &= \mathrm{e}^{-\mathrm{i}\omega_{0j}t - \mathrm{i}n_{0j}\zeta + \mathrm{i}m_{0j}\vartheta}a_{0j}(t)\sum_h \mathrm{e}^{\mathrm{i}h\vartheta}\Phi_{0j}(n_{0j}q - m_{0j} - h),\\ (e/T_\mathrm{e})\delta\phi_{1j} &= \mathrm{e}^{-\mathrm{i}\omega_{1j}t - \mathrm{i}n_{1j}\zeta + \mathrm{i}m_{1j}\vartheta}a_{1j}(t)\sum_h \mathrm{e}^{\mathrm{i}h\vartheta}\Phi_{1j}(n_{1j}q - m_{1j} - h),\end{aligned} \tag{7}$$

for the (m_{0j}, n_{0j}) and (m_{1j}, n_{1j}) ETG modes, consistent with equation (4), with $\omega_\ell = \omega_{0j} - \omega_{1j}$. Given equation (6), the quasi-mode structure in ballooning space can be conveniently represented as superposition of *partial* quasi-modes,

$$a_\ell\hat{\Phi}_\ell(\theta) = \sum_j a_{\ell j}\hat{\Phi}_{\ell j}(\theta) \tag{8}$$

with

$$\hat{L}_\ell\hat{\Phi}_{\ell j}(\theta) = \frac{\pi\tau}{k_{\vartheta 0j}^{1/2}k_{\vartheta 1j}^{1/2}k_{\vartheta\ell}}\sum_l 2\pi l\,\delta(\theta - 2\pi l)\int_{-\infty}^{+\infty}(1 + s^2\eta^2) \times \left(k_{\vartheta 1j}^2\hat{\Phi}_{0j}(\eta_l)\hat{\Phi}_{1j}^*(\eta) - k_{\vartheta 0j}^2\hat{\Phi}_{0j}(\eta)\hat{\Phi}_{1j}^*(\eta_l)\right)\mathrm{d}\eta \tag{9}$$

and [1,2]

$$\partial_t a_{\ell j}(t) = 2s\frac{\hat{\alpha}_\mathrm{e}}{\tau}k_{\vartheta 0j}^{3/2}k_{\vartheta 1j}^{3/2}k_{\vartheta\ell}a_{0j}(t)a_{1j}^*(t) \tag{10}$$

since $|\omega_\ell^{-1}\partial_t| \gg 1$ within the frequency ordering adopted here. Considering $\hat{L}_\ell \simeq \tau$, each of the partial quasi-modes feeds back on the primary ETG modes as follows:

$$
\begin{aligned}
(-\mathrm{i}\omega_{0j} + \partial_t)\hat{L}_0 a_{0j}(t)\hat{\Phi}_{0j}(\theta) = & -s\pi\hat{\alpha}_\mathrm{e} a_{1j}(t) a_{\ell j}(t) \frac{k_{\vartheta 0j}^{1/2} k_{\vartheta 1j}^{1/2}}{k_{\vartheta\ell}} \sum_l 4\pi^2 l^2 \\
& \times \hat{\Phi}_{1j}(\theta - 2\pi l)\left[k_{\vartheta 1j}^2 + s^2 k_{\vartheta 1j}^2(\theta - 2\pi l)^2 - 4\pi^2 l^2 s^2 k_{\vartheta 0j}^2\right] \\
& \times \int_{-\infty}^{+\infty} (1 + s^2\eta^2)\left(k_{\vartheta 1j}^2 \hat{\Phi}_{0j}(\eta_l)\hat{\Phi}_{1j}^*(\eta) - k_{\vartheta 0j}^2 \hat{\Phi}_{0j}(\eta)\hat{\Phi}_{1j}^*(\eta_l)\right)\mathrm{d}\eta \\
& + s\pi\hat{\alpha}_\mathrm{e} a_{0j+1}(t) a_{\ell j+1}^*(t) \frac{k_{\vartheta 0j}^{1/2} k_{\vartheta 0j+1}^{1/2}}{k_{\vartheta\ell}} \sum_l 4\pi^2 l^2 \hat{\Phi}_{0j+1}(\theta + 2\pi l) \\
& \times \left[k_{\vartheta 0j+1}^2 + s^2 k_{\vartheta 0j+1}^2(\theta + 2\pi l)^2 - 4\pi^2 l^2 s^2 k_{\vartheta 0j}^2\right] \int_{-\infty}^{+\infty} (1 + s^2\eta^2) \\
& \times \left(k_{\vartheta 0j}^2 \hat{\Phi}_{0j+1}^*(\eta_l)\hat{\Phi}_{0j}(\eta) - k_{\vartheta 0j+1}^2 \hat{\Phi}_{0j+1}^*(\eta)\hat{\Phi}_{0j}(\eta_l)\right)\mathrm{d}\eta.
\end{aligned} \tag{11}
$$

Projecting on $\hat{\Phi}_{0j}(\theta)$, we easily obtain

$$
\begin{aligned}
(\partial_t - \gamma_{0j}) a_{0j}(t) = & -\frac{s\pi\hat{\alpha}_\mathrm{e}/\tau}{\int_{-\infty}^{\infty}|\hat{\Phi}_{0j}(\theta)|^2\mathrm{d}\theta} a_{0j-1}(t) a_{\ell j}(t) \frac{k_{\vartheta 0j}^{1/2} k_{\vartheta 0j-1}^{1/2}}{k_{\vartheta\ell}} \sum_l 4\pi^2 l^2 \\
& \times \left(k_{\vartheta 0j-1}^2 D_{0j,l}^* - 4\pi^2 l^2 s^2 k_{\vartheta 0j}^2 E_{0j,l}^*\right)\left(k_{\vartheta 0j-1}^2 C_{0j,l} - k_{\vartheta 0j}^2 D_{0j,l}\right) \\
& + \frac{s\pi\hat{\alpha}_\mathrm{e}/\tau}{\int_{-\infty}^{\infty}|\hat{\Phi}_{0j}(\theta)|^2\mathrm{d}\theta} a_{0j+1}(t) a_{\ell j+1}^*(t) \frac{k_{\vartheta 0j}^{1/2} k_{\vartheta 0j+1}^{1/2}}{k_{\vartheta\ell}} \sum_l 4\pi^2 l^2 \\
& \times \left(k_{\vartheta 0j+1}^2 D_{0j+1,l} - 4\pi^2 l^2 s^2 k_{\vartheta 0j}^2 E_{0j+1,l}\right) \\
& \times \left(k_{\vartheta 0j}^2 C_{0j+1,l}^* - k_{\vartheta 0j+1}^2 D_{0j+1,l}^*\right),
\end{aligned} \tag{12}
$$

where γ_{0j} is the linear growth rate of the $(m_0 + jm_\ell, n_0 + jn_\ell)$ ETG mode, with $j = 0, \pm 1, \pm 2, \ldots,$

$$
\begin{aligned}
C_{0j,l} &= \int_{-\infty}^{\infty} (1 + s^2\theta^2)\hat{\Phi}_{0j}(\theta_l)\hat{\Phi}_{0j-1}^*(\theta)\mathrm{d}\theta, \\
D_{0j,l} &= \int_{-\infty}^{\infty} (1 + s^2\theta^2)\hat{\Phi}_{0j}(\theta)\hat{\Phi}_{0j-1}^*(\theta_l)\mathrm{d}\theta, \\
E_{0j,l} &= \int_{-\infty}^{\infty} \hat{\Phi}_{0j}(\theta_l)\hat{\Phi}_{0j-1}^*(\theta)\mathrm{d}\theta,
\end{aligned} \tag{13}
$$

are structure constants, and we have used the fact that $\omega_{0j}\partial_{\omega_{0j}} D(\omega_{0j}) \simeq \tau$, with

$$
D(\omega_{0j}) = \left(\int_{-\infty}^{\infty} |\hat{\Phi}_{0j}(\theta)|^2\mathrm{d}\theta\right)^{-1} \int_{-\infty}^{\infty} \hat{\Phi}_{0j}^*(\theta)\hat{L}_0\hat{\Phi}_{0j}(\theta)\mathrm{d}\theta. \tag{14}
$$

Equations (10) and (12) govern the nonlinear spectral transfer for ETG, due to nonlinear toroidal mode coupling. They are already in a fairly simple form and could be taken as a nonlinear ordinary differential equation system to solve for ETG and quasi-mode amplitudes, at least numerically. We can, however, further simplify the structure of these equations using the fact that, typically, $n_\ell \ll |\gamma/\omega| n_0$, i.e. by taking the continuum limit, and neglecting the weak variability of the parallel mode structure, $\hat{\Phi}_{0j}(\theta)$, with the mode number. Furthermore we assume that $\hat{\Phi}_{0j}(\theta)$ is nearly real, consistent with $|\gamma/\omega| \ll 1$. Note that the latter assumption is equivalent to neglecting the nonlinear frequency shift, which is instead accounted for in equation (12). In this way, equation (12) becomes [1,2]

$$(\partial_t - 2\gamma_{0j} + 2\gamma_{\mathrm{NL}j})I_{0j} + \partial_{n_{0j}}(v_{n_{0j}} I_{0j}) = 0, \tag{15}$$

where $I_{0j} = \bar{a}_{0j}^2/2$, $a_{0j} = \mathrm{e}^{\mathrm{i}\varphi_{0j}}\bar{a}_{0j}$, $a_{\ell j} = \mathrm{e}^{\mathrm{i}\varphi_{\ell j}}\bar{a}_{\ell j}$, $\varphi_{\ell j} = \varphi_{0j} - \varphi_{0j-1}$, $k_{\vartheta 0j} = n_{0j}q_0/r_0$ and

$$\begin{aligned} v_{n_{0j}} &= -\frac{4\pi s\hat{\alpha}_e/\tau}{\|\hat{\Phi}_{0j}\|^2} k_{\vartheta 0j}^4 n_\ell \sum_l 4\pi^2 l^2 (4\pi^2 l^2 s^2 E_{0j,l} - C_{0j,l}) C_{0j,l}\bar{a}_{\ell j}, \\ \gamma_{\mathrm{NL}j} &= \frac{4\pi s\hat{\alpha}_e/\tau}{\|\hat{\Phi}_{0j}\|^2} k_{\vartheta 0j}^3 k_{\vartheta\ell} \sum_l 4\pi^2 l^2 (4\pi^2 l^2 s^2 E_{0j,l} + C_{0j,l}) C_{0j,l}\bar{a}_{\ell j} \end{aligned} \tag{16}$$

with $\|\hat{\Phi}_{0j}\|^2 = \int_{-\infty}^{\infty} |\hat{\Phi}_{0j}(\theta)|^2 \mathrm{d}\theta$. Note that, here, we have considered $C_{0j,l} \simeq D_{0j,l}$ and $E_{0j,l}$ as being real, consistent with the present assumptions. Meanwhile, equation (10) can be rewritten as

$$(\partial_t + \gamma_{\ell j})\bar{a}_{\ell j} = 4s(\hat{\alpha}_e/\tau)k_{\vartheta 0j}^3 k_{\vartheta\ell} I_{0j}. \tag{17}$$

Here, with respect to equation (10), we have considered a finite quasi-mode damping $\gamma_{\ell j}$. Equations (15) and (17) can be combined into the energy conservation,

$$(\partial_t + 2\gamma_{\ell j})I_{\ell j} + (\partial_t - 2\gamma_{0j})I_{0j} + \partial_{n_{0j}}(v_{n_{0j}} I_{0j}) = 0 \tag{18}$$

with

$$I_{\ell j} = 2\pi \|\hat{\Phi}_{0j}\|^{-2} (\bar{a}_{\ell j}^2/2) \sum_l 4\pi^2 l^2 (4\pi^2 l^2 s^2 E_{0j,l} + C_{0j,l}) C_{0j,l}. \tag{19}$$

In contrast to equations (10) and (12), (15) and (17) are not characterized by a cut-off at large quasi-mode toroidal numbers, and the nonlinear coupling coefficients increase linearly with n_ℓ. This is a by-product of the continuum limit, based on $n_\ell \ll |\gamma/\omega| n_0$, and must be kept in mind when using these simplified equations. In the early nonlinear phase, the dominant effect of nonlinear toroidal mode coupling is to produce an *inverse cascade* of the ETG spectrum, with the quasi-modes acting as mediators but scarcely contributing to the energy balance [1,2]. The reason is that the typical width of the ETG spectrum in this phase is $\Delta n/n_{0j} \lesssim |\gamma/\omega|$, i.e. $\partial_{n_{0j}} \approx \Delta n^{-1} \gtrsim \mathrm{O}(n_{0j}^{-3/4})$, and the nonlinear damping term can be neglected in equation (15) for sufficiently short times, as assumed in [1,2]. On longer time scales, the energy content of quasi-modes is important for ensuring energy conservation, as clearly expressed in equation (18). Note, also, that equations (15) to (19) consider the incoherent effect of all partial quasi-modes with one single n_ℓ value (see equation (8)), due to the random phase approximation. However, scattering off a given partial quasi-mode generates secondary ETG modes, as shown in equation (12). Thus, the ETG component of the fluctuation spectrum is composed of both *spontaneous* and *stimulated* plasmons: the former are due to the usual linear excitation and the latter correspond to secondary ETG mode generation via scattering off quasi modes, as in the coherent process in laser light emission. For this reason, we must account for an effective number of coherent states, $N_{c\ell} = \nu_{c\ell}\Delta n/n_\ell$, in the partial quasi-mode decomposition of equation (8), where $(n_{\mathrm{c.o.}}/\Delta n) \lesssim \nu_{c\ell} \leqslant 1$, the lower/upper bounds corresponding, respectively, to the fully incoherent/coherent cases. Furthermore, we must consider the simultaneous effect of all possible n_ℓ quasi-modes, whose *density of states* is $\sigma_\ell = (\Delta n/n_\ell)$ ($\Delta n < n_{\ell\mathrm{m}}$, see discussion following equation (3)). Rather than rewriting equations (15) to (19) to account for these facts, we will simply assume an implicit $\sum_{n_\ell} \simeq (1/2)\int_{n_{\mathrm{c.o.}}}^{\Delta n} N_{c\ell}\sigma_\ell n_\ell^{-1} \mathrm{d}n_\ell = (\nu_{c\ell}/2)\Delta n^2 \int_{n_{\mathrm{c.o.}}}^{\Delta n} n_\ell^{-3}\mathrm{d}n_\ell$ on the right-hand side in equation (16) and consider $\sum_{n_\ell}(\partial_t + 2\gamma_{\ell j})I_{\ell j}$ rather than $(\partial_t + 2\gamma_{\ell j})I_{\ell j}$ in equation (18).

An important implication of equations (15) and (17), with the expression of nonlinear coupling coefficients given in equation (16) in terms of the structure constants of equation (13), is that nonlinear toroidal coupling crucially depends on the toroidal nature of the plasma equilibrium via the parallel mode structure. A fundamental result is that nonlinear interactions are maximized for *moderately ballooning* mode structures: for strongly ballooning modes, nonlinear couplings become weaker due to the decreasing overlap of mode structures in ballooning space; conversely, for weakly ballooning modes, it is the inverse cascade characteristic rate which is reduced. The inverse cascade process is also regulated by the magnetic shear. In fact, equation (16) shows that cascading towards a longer wavelength is possible for finite s^2 only. For vanishing magnetic shear, the nonlinear toroidal coupling vanishes as well, and the present paradigm for ETG nonlinear dynamics is not applicable. All these considerations can be put on a more quantitative basis by estimating the normalized nonlinear coupling constant, defined as $\kappa_{\ell j} = \|\hat{\Phi}_{0j}\|^{-4}\sum_l 4\pi^2 l^2 s^2(4\pi^2 l^2 s^2 E_{0j,l} - C_{0j,l})C_{0j,l}$, for $l = 1$ and using equation (13), with a trial mode structure $\hat{\Phi}_{0j} = \exp(-\theta^2/\Delta\theta^2)$. We find

$$\kappa_{\ell j} \simeq 4\pi^2 s^2[4\pi^2 s^2 - (1 + \pi^2 s^2(1 + \Delta\theta^2/4\pi^2))] \times(1 + \pi^2 s^2(1 + \Delta\theta^2/4\pi^2)) \exp(-4\pi^2/\Delta\theta^2). \quad (20)$$

3. Generation of ZFs

In this section, we demonstrate that spontaneous excitation of ZFs by modulational instability [3, 4] of ETG turbulence occurs on a typically longer time scale than does the inverse cascade due to nonlinear toroidal mode couplings, described in section 4, and, thus, that it is a process consistently neglected therein. We strictly follow [4] and assume that one coherent toroidal ETG mode $\delta\phi_0$, with toroidal mode number n_0, interacts with a zonal scalar potential $\delta\phi_z$, generating sideband modes $\delta\phi_+ \Leftarrow \delta\phi_z \times \delta\phi_0$ and $\delta\phi_- \Leftarrow \delta\phi_z \times \delta\phi_0^*$. Due to the large ion Larmor radius, the ion adiabatic response is a good assumption for both ZFs and sidebands, which are, thus, described by equation (1). In the ballooning space, sideband excitation by ZFs is described by

$$\partial_t \hat{L}_\pm \hat{\Phi}_\pm(\theta) a_\pm = \mp\hat{\alpha}_e k_{\vartheta 0} k_z \left[k_{\vartheta 0}^2(1 + s^2\theta^2) - k_z^2\right] a_z \begin{pmatrix} a_0\hat{\Phi}_0(\theta) \\ a_0^*\hat{\Phi}_0^*(\theta)\end{pmatrix} \quad (21)$$

with the same notation as in section 2 and $\boldsymbol{k_z} = k_z\nabla r/|\nabla r|$ the ZF wave-vector. After projecting on $\hat{\Phi}_0(\theta)$, one finally obtains [9]

$$a_\pm = \mp\frac{\hat{\alpha}_e}{\tau}\frac{k_{\vartheta 0}k_z(\langle\langle k_{\perp 0}^2\rangle\rangle - k_z^2)}{(\Gamma_z + \gamma_d \mp i\Delta)} a_z \begin{pmatrix} a_0 \\ a_0^* \end{pmatrix}. \quad (22)$$

Here, $\langle\langle k_{\perp 0}^2\rangle\rangle = k_{\vartheta 0}^2\int_{-\infty}^{\infty}(1 + s^2\theta^2)|\hat{\Phi}_0|^2 d\theta / \int_{-\infty}^{\infty}|\hat{\Phi}_0|^2 d\theta$, $(-i\omega D)_\pm \simeq \omega_0\partial_{\omega_0}D(-i\omega_z + \gamma_d \mp i\Delta) \simeq \tau(\Gamma_z + \gamma_d \mp i\Delta)$, γ_d is the sideband damping, $\omega_\pm = \omega_z \pm \omega_0$, $\Gamma_z = -i\omega_z$ is the ZF modulational instability growth rate and $\Delta = (k_z^2/2)(\partial^2 D/\partial k_r^2)(\partial D/\partial\omega_0)^{-1} \simeq \omega_0(k_z^2/2\tau)(\partial^2 D/\partial k_r^2)$ [4]. Meanwhile, the evolution equation for the ZF amplitude is straightforwardly obtained from equation (1) in the following form:

$$(\partial_t + \gamma_z)\tau a_z = -2\pi\hat{\alpha}_e k_{\vartheta 0}\sum_l e^{i2\pi lnq}(k_z + 2\pi lnq')^2\int_{-\infty}^{\infty}\hat{\Phi}_0(\theta)\hat{\Phi}_0^*(\theta_l) \times[(k_z + 2\pi lnq' + 2nq'\theta_l)a_0^* a_+ - (k_z + 2\pi lnq' - 2nq'\theta)a_0 a_-]d\theta. \quad (23)$$

Here, τ on the left-hand side is the approximate value of the ETG-induced ZF polarizability and γ_z is the ZF collisional damping [15,16], whereas the right-hand side includes both meso-scale

($\mathrm{O}(k_z^{-1})$) variations of the zonal scalar potential, $e\delta\phi_z/T_\mathrm{e} = a_z \mathrm{e}^{-\mathrm{i}\omega_z t}$, and fine radial scales that trace back to the $\mathrm{O}[(nq')^{-1}]$ structures of ETG and sidebands. Such fine radial structures are inefficient in spontaneously exciting ZFs [4] (see also equations (22) and (25)); thus, we may consider the spatial average of equation (23) and describe the meso-scale excitation of ZF via [4, 9]

$$(\partial_t + \gamma_z)\tau a_z = -2\pi\hat{\alpha}_\mathrm{e} k_{\vartheta 0} k_z^3 \|\hat{\Phi}_0\|^2 \left(a_0^* a_+ - a_0 a_-\right). \tag{24}$$

Equations (22) and (24) readily describe spontaneous ZF excitation by ETG turbulence and clearly illustrate the crucial differences of this case with respect to the analogue of ZF excitation by ITG [9]. More precisely, equation (22) shows that the sideband amplitude is weaker in the ETG case than in the ITG case by a factor $\mathrm{O}(k_\perp^2\rho_\mathrm{e}^2)$ since the ETG sideband excitation by ZFs is due to Hasegawa–Mima [17] nonlinear interactions (see equation (1)) rather than by the $\boldsymbol{E}\times\boldsymbol{B}$ nonlinearity as in the ITG case [4]. Meanwhile, equation (24) indicates that the ETG-induced ZF polarizability, $\simeq \tau$, is higher than in the ITG case by a factor $\mathrm{O}(k_\perp^{-2}\rho_\mathrm{e}^{-2})$. As a result, the spontaneous ZF generation rate by ETG turbulence is reduced by at least $\mathrm{O}(k_\perp^2\rho_\mathrm{e}^2)$ with respect to the ITG case [9]. In both equations (22) and (24), the reason for different sideband/ZF behaviours in the ETG and ITG cases is that ions respond adiabatically to ZF modulations in ETG turbulence because of their large orbits, whereas electrons do not respond at all to ZFs in ITG turbulence. Combining equation (22) and (24), the ZF modulational instability growth rate, Γ_z, is obtained as [4]

$$\frac{(\Gamma_z+\gamma_z)}{(\Gamma_z+\gamma_\mathrm{d})}[(\Gamma_z+\gamma_\mathrm{d})^2+\Delta^2] = 4\pi\frac{\hat{\alpha}_\mathrm{e}^2}{\tau^2}k_{\vartheta 0}^2 k_z^4(\langle\langle k_{\perp 0}^2\rangle\rangle - k_z^2)\sum_{n_0}\|\hat{\Phi}_0\|^2|a_0|^2, \tag{25}$$

where $\sum_{n_0}$ on the right-hand side accounts for the incoherent effect of all toroidal mode numbers when computing the ZF growth rate and, by the same arguments as those following equation (19) in section 2, can be estimated as $\sum_{n_0} \approx \Delta n/n_{\mathrm{c.o.}}$. Thus, the fastest growing ZF is characterized by $k_z^2 = \frac{2}{3}\langle\langle k_{\perp 0}^2\rangle\rangle$. For strong ZF excitation, $\Gamma_z \gg \gamma_z, \gamma_\mathrm{d}, \Delta$, equation (25) yields a linear Γ_z scaling with the ETG amplitude, i.e.

$$\Gamma_z \simeq 2\pi^{1/2}(\hat{\alpha}_\mathrm{e}/\tau)|k_{\vartheta 0}|k_z^2(\langle\langle k_{\perp 0}^2\rangle\rangle - k_z^2)^{1/2}\left(\sum_{n_0}\|\hat{\Phi}_0\|^2|a_0|^2\right)^{1/2}, \tag{26}$$

confirming the reduction by a factor $\mathrm{O}(k_\perp^2\rho_\mathrm{e}^2)$ with respect to the ITG case [4], as anticipated above. However, this condition of strong ZF excitation is unlikely to occur, since the validity of equation (26) would require, *ceteris paribus*, an ETG amplitude larger than the corresponding ITG level by a factor $\mathrm{O}(k_\perp^{-2}\rho_\mathrm{e}^{-2})$. That $\Gamma_z \gg \gamma_z, \gamma_\mathrm{d}, \Delta$ is inconsistent with the present ordering is readily verified. In fact $\Delta/\omega_0 \approx k_z^2/k_{\vartheta 0}^2(\omega_\mathrm{d}/\omega_0) \approx \gamma_0/\omega_0$, with ω_d the magnetic drift frequency, and $\gamma_\mathrm{d}/\omega_0 \approx k_z/k_{\vartheta 0}(\gamma_0/\omega_0) \approx \gamma_0/\omega_0$. The applicability condition of equation (26) is then $(\sum_{n_0}|e\delta\phi_0/T_\mathrm{e}|^2)^{1/2} \gg |\gamma_0/\omega_0||k_{\vartheta 0}\rho_\mathrm{e}|^{-4}|\omega_0/\omega_\mathrm{ce}|$, i.e. an unrealistically high fluctuation level, not only in contrast with numerical simulations [1, 2] but conflicting with the nonlinear gyrokinetic equation ordering itself [10]; thus, a more realistic regime to consider in the ETG case is $\gamma_\mathrm{d} \approx \Delta \gg \Gamma_z \gg \gamma_z$, for which equation (25) yields

$$\Gamma_z = 4\pi\frac{\hat{\alpha}_\mathrm{e}^2}{\tau^2}\frac{\gamma_\mathrm{d}}{\gamma_\mathrm{d}^2+\Delta^2}k_{\vartheta 0}^2 k_z^4(\langle\langle k_{\perp 0}^2\rangle\rangle - k_z^2)\sum_{n_0}\|\hat{\Phi}_0\|^2|a_0|^2. \tag{27}$$

Therefore, the characteristic condition of ETG modulational instability is much weaker than that of ITG turbulence and similar in nature to that of drift wave plasmons, with the growth rate $\propto (\mathrm{e}\delta\phi_0/T_\mathrm{e})^2$ [4]. Meanwhile, the fastest growing ZF is characterized by $\frac{1}{3}\langle\langle k_{\perp 0}^2\rangle\rangle \lesssim k_z^2 \lesssim \frac{3}{5}\langle\langle k_{\perp 0}^2\rangle\rangle$.

In order to derive the typical relative ordering of the ETG inverse cascade and spontaneous ZF excitation, we must estimate the inverse cascade characteristic rate, $\tau_{\mathrm{NL},c}^{-1}$ from equation (15). In the early nonlinear phase, when $I_{0j} \propto \mathrm{e}^{2\gamma_{0j}t}$, equation (17) gives $\bar{a}_{\ell j} \simeq 2s\hat{\alpha}_{\mathrm{e}}/(\tau\gamma_{0j})k_{\vartheta 0j}^3 k_{\vartheta\ell} I_{0j}$. Substituting into equation (15) and summing on all quasi-modes, with density of states $\sigma_\ell = \Delta n/n_\ell$ and an effective number of coherent states $N_{\mathrm{c}\ell} = \nu_{\mathrm{c}\ell}\Delta n/n_\ell$ (see discussion following equation (19)), we have

$$\tau_{\mathrm{NL},c}^{-1} \simeq \frac{4\pi s^2\hat{\alpha}_{\mathrm{e}}^2}{\gamma_{0j}\tau^2}\frac{\partial}{\partial n_{0j}}\left(k_{\vartheta 0j}^8\kappa_{\ell j}I_{0j}\|\hat{\Phi}_{0j}\|^2\nu_{\mathrm{c}\ell}\int_{n_{\mathrm{c.o.}}}^{\Delta n}\frac{\Delta n^2}{n_{0j}n_\ell}\mathrm{d}n_\ell\right), \tag{28}$$

where the nonlinear coupling constant, $\kappa_{\ell j}$, is given in equation (20) and above. Estimating $\partial_{n_{0j}} \approx \Delta n^{-1}$, a comparison of equation (27) with equation (28) gives

$$\tau_{\mathrm{NL},c}^{-1}\Gamma_z^{-1} \gtrsim \nu_{\mathrm{c}\ell}\kappa_{\ell j}\left(n_{\mathrm{c.o.}}\Delta n/n_{0j}\right)\ln(\Delta n/n_{\mathrm{c.o.}}). \tag{29}$$

For typical conditions, $n_{\mathrm{c.o.}} \lesssim \Delta n = \mathrm{O}(n^{1/2})$ and $\nu_{\mathrm{c}\ell} = \mathrm{O}(1)$, equation (29) suggests that there should be a very weak residual dependence of $\tau_{\mathrm{NL},c}^{-1}\Gamma_z^{-1}$ on n (or system size). Meanwhile, as suggested by equation (20), we have, typically, $\kappa_{\ell j} \gg 1$ and, hence, $\tau_{\mathrm{NL},c}\Gamma_z \lesssim 10^{-1}$ with the present ordering. Thus, we expect to observe ZF effects on ETG nonlinear dynamics on time scales longer than those typical of the inverse spectral cascade, which dominates the saturation process; consistent with numerical simulations that show a moderate radial modulation of the ETG elongated eddies—the *streamers*—at the end of the longest simulation runs [2]. This estimate may change for the fully incoherent case, $\nu_{\mathrm{c}\ell} \approx (n_{\mathrm{c.o.}}/\Delta n) \ll 1$, i.e. for moderate $n_{\mathrm{c.o.}}$, which implies q_0 to be a very low order rational number, and/or for small shear, i.e. for small $\kappa_{\ell j}$, as shown in equation (20). Under such conditions, the role of ZF could be crucial in the nonlinear ETG evolution on the inverse spectral cascade time scale.

4. ETG nonlinear saturation

In the discussion following equation (3), we emphasized that ETG fluctuations are composed of two plasmon distributions: the high-n linearly driven ETG mode which becomes marginally stable at $n \lesssim n_{\mathrm{MS}}$, with $n_{\mathrm{MS}} = \mathrm{O}(n_0^{3/4})$ [1,2], and the low-n_ℓ forced quasi-modes, characterized by an upper cutoff at $n_{\ell\mathrm{m}} \simeq (\gamma/\omega)n = \mathrm{O}(n_0^{3/4})$, i.e. $n_{\mathrm{MS}} \approx n_{\ell\mathrm{m}}$. Here, n_0 is the typical toroidal mode number of the most unstable linear ETG. In the following, we argue that this point has important consequences for the shape of the saturated ETG spectrum.

Given the proof (see section 3) that the ZF generation rate is much smaller than the spectral transfer rate due to nonlinear toroidal couplings, the ETG saturation proceeds via cascading of linearly unstable modes to longer wavelength and more weakly growing ETG modes by scattering off the quasi-modes. Equation (15) predicts that the saturated ETG spectrum peaks where the net nonlinear growth rate vanishes. Thus, the spectral maximum must be located near $n_{\mathrm{MS}} = \mathrm{O}(n_0^{3/4})$, i.e. $k_{\vartheta,\max}\rho_{\mathrm{e}} \approx n_0^{-1/2} = \mathrm{O}(10^{-1})$, with a slight shift towards shorter wavelengths due to the effect of nonlinear damping. Since $k_{\vartheta,\max}\rho_{\mathrm{e}} \approx n_0^{-1/2}$ for the high-n ETG plasmon distribution and the upper cutoff of the quasi-mode spectrum is at $n_{\ell\mathrm{m}} \simeq (\gamma/\omega)n_0 = \mathrm{O}(n_0^{3/4})$, i.e. $k_{\vartheta,\ell\mathrm{m}}\rho_{\mathrm{e}} \approx n_0^{-1/2} \approx k_{\vartheta,\max}\rho_{\mathrm{e}}$, the two plasmon distributions constituting the ETG fluctuation spectrum have significant overlap, and only one spectral peak may be expected, corresponding to the high-n ETG plasmon distribution maximum.

Besides estimating the location of the ETG saturated spectrum peak, as predicted by equation (15), we can also estimate its width. In fact, according to equation (17), when the quasi-mode damping is very small, the quasi-mode amplitude can continue growing even after the inverse cascade spectral transfer rate locally (in n-space) balances the linear growth rate. When this happens, the ETG spectrum is *depleted* because the inverse cascade process pumps energy out faster than linear excitation pumps energy in a given k_ϑ. The necessary condition for this is the (exponentially) weak quasi-mode damping, i.e. $\gamma \simeq (\gamma/\omega)(n/n_0)\omega_0 \gtrsim (\gamma/\omega)^{-1}|k_{\|\ell}|v_e \simeq (\gamma/\omega)^{-1}(n_\ell/n)|k_\||v_e$. Thus, we expect that the large k_ϑ linear ETG spectrum, with $k_\vartheta^2/k_{\vartheta 0}^2 \gtrsim (\gamma/\omega)^{-2}(n_\ell/n_0)(|k_\||v_e/|\omega_0|) \approx (|k_\||v_e/|\omega_0|) \approx n_0^{-1/4}$, is significantly depleted.

The dynamics of nonlinear ETG saturation and the qualitative features of the ETG saturated spectrum, as outlined in this section, are consistent with the picture of [2] and is summarized in figure 1 therein. Numerical solutions of equations (15) to (17), in principle, would allow us to exactly compute the ETG spectrum at saturation and its dynamic formation. However, this is out of the scope of the present work. In future studies, we will also examine the potentially important role of the nonlinearly generated $(n, m) = (0, 1)$ quasi-mode, which may modify the ETG growth/damping rate by altering the potential well structure parallel to the magnetic field line.

5. Discussion and Conclusions

The different nonlinear dynamics of ETG and ITG turbulence are essentially due to the different responses of, respectively, ions and electrons to ZF fluctuations. Drift wave–ZF interactions regulate the ITG nonlinear dynamics, whereas the nonlocal inverse spectral cascade via scattering off driven quasi-modes determines the ETG saturation [1,2]. However, both ZF spontaneous generation and inverse cascading due to nonlinear toroidal coupling are dynamic phenomena of ETG, ITG and, more generally, drift wave turbulence, which may take place on different nonlinear time scales. In fact, on long time scales the ITG saturated spectrum is characterized by a peak downshift towards longer wavelengths, and the ETG modes show a radial modulation of the extended streamer structures by the ZF [2]. The theoretical framework developed in this work, under fairly general assumptions, is consistent with these observations based on numerical simulation results [1, 2]. The detailed nature of the nonlinear toroidal mode-coupling process is shown to depend critically on the ballooning-mode structures in toroidal geometries and, as such, represents a new paradigm for the spectral cascade of plasma turbulence in toroidal systems.

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