LETTERS

The purpose of this Letters section is to provide rapid dissemination of important new results in the fields regularly covered by Physics of Plasmas. Results of extended research should not be presented as a series of letters in place of comprehensive articles. Letters cannot exceed four printed pages in length, including space allowed for title, figures, tables, references and an abstract limited to about 100 words. ***There is a three-month time limit, from date of receipt to acceptance, for processing Letter manuscripts. Authors must also submit a brief statement justifying rapid publication in the Letters section.***

On resonant heating below the cyclotron frequency

Liu Chen
Department of Physics and Astronomy, University of California, Irvine, California 92697

Zhihong Lin and Roscoe White
Plasma Physics Laboratory, Princeton University, P.O. Box 451, Princeton, New Jersey 08543

(Received 15 June 2001; accepted 8 August 2001)

Resonant heating of particles by electrostatic and Alfvén waves propagating in a confining uniform magnetic field is examined. It is shown that, with a sufficiently large wave amplitude, significant perpendicular stochastic heating can be obtained with wave frequency at a fraction of the cyclotron frequency. This result may have relevance for the heating of ions in the solar corona, and is a generic phenomenon, independent of the type of wave considered.
[DOI: 10.1063/1.1406939]

Resonant heating of particles in a confining magnetic field has been examined by many authors and is of importance in the heating of magnetically confined laboratory as well as extraterrestrial plasmas.[1–7] To our knowledge, however, a breaking of the invariance of the magnetic moment at frequencies at a fraction of the cyclotron frequency has never been reported in a theoretical or experimental work. In this letter, we wish to demonstrate that, at sufficiently large wave amplitudes, low-frequency wave heating is indeed possible. Consider first the simple model problem of a particle gyrating in a constant magnetic field $\mathbf{B}_0=B_0\hat{z}$ acted upon by an electrostatic plane wave propagating perpendicularly to the magnetic field $\mathbf{B}_0$. Physically, this may, for example, correspond to the case of electron heating by lower-hybrid waves in high density/low magnetic field plasmas.

The Hamiltonian for this system is $H=(\mathbf{p}-\mathbf{A})^2/2+\Phi(x,t)$ with the magnetic field given by the vector potential $\mathbf{A}=-B_0y\hat{x}$. Take the units of time to be given by Ω_c, the cyclotron frequency, let the electrostatic wave be given by a single harmonic, $\Phi=\Phi_0\cos(kx-\omega t)$, and set the velocity parallel to $\mathbf{B}_0$ (a constant of the motion) to zero, $v_z=0$. The three dimensionless parameters characterizing the heating problem are $k\rho$, giving the ratio of cyclotron radius to wave length, with $\rho=v/\Omega_c$ the cyclotron radius, $k^2\Phi_0=|\mathbf{k}\cdot\mathbf{x}_g|$, the nonlinearity parameter with $\mathbf{x}_g$ being the wave-induced guiding-center polarization-drift displacement, and ω, the ratio of the wave frequency to the cyclotron frequency.

The equations of motion become $\dot{v}_x=v_y+k\Phi_0\sin(kx-\omega t)$, $v_y=-x+x_0$, giving

$$\frac{d^2x}{dt^2}+x=x_0+k\Phi_0\sin(kx-\omega t). \tag{1}$$

First, it is illuminating to consider Eq. (1) for $s\equiv k(x-x_0)\ll 1$. Letting $2T=kx_0-\omega t$ and keeping only the lowest order in s (i.e., linearization), we have

$$\frac{d^2s}{dT^2}+\left[\frac{4}{\omega^2}-\frac{4k^2\Phi_0}{\omega^2}\cos(2T)\right]s=\frac{4k^2\Phi_0}{\omega^2}\sin(2T); \tag{2}$$

i.e., a driven Mathieu equation with unstable solutions for $\omega\simeq 2/q$ with q integer. Of course this equation is valid only for small s, but it indicates the existence of large amplitude solutions for these values of ω. This response is simply interpretable as due to resonance consisting of an integer number of cyclotron oscillations within one wave oscillation. The fact that there are many such resonances implies that, at large amplitudes, stochastic threshold should be attained, permitting heating.

Now consider a Poincaré section of $k\rho$, $\psi=kx-\omega t$, by taking points when $v_y=0,\dot{v}_y>0$. We find that resonances exist for $\omega=2/q$ for all integer q, associated with the unstable domains of the associated Mathieu equation. Secularities are not found at fractional frequencies in the standard Hamiltonian analysis[1] because the wave field is considered only to first order. A higher-order analysis demonstrates their existence and, at small amplitudes, they can be found analytically. Physically, for $k^2\Phi_0\sim 1$, $\mathbf{k}\cdot\mathbf{x}_g$ contains finite higher harmonics in ω and, thus, renders nonlinear resonances with cyclotron motion Ω_c possible.

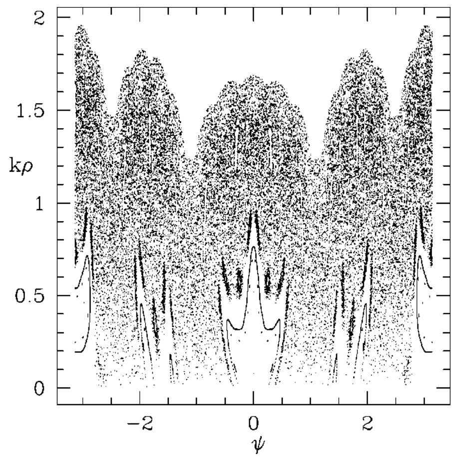

FIG. 1. Poincaré, $k^2\Phi_0=0.77$, $\omega=1/4$.

Now investigate the approach to chaos and the extent of the chaotic domain, which limits the possible heating obtained. Figure 1 shows an example of the resonances and the extent of the stochastic domain for $\omega=1/4$, with $k^2\Phi_0=0.77$, bounded by good Kolmogorov–Arnold–Moser (KAM) surfaces at large $k\rho$. The initial particle distribution was random with $k\rho<0.1$. The magnetic moment again becomes an adiabatic invariant for very large energy.[3] Heating of an initially cold distribution proceeds to the maximum limit given by good KAM surfaces in a rather short time, on the order of one to two hundred cyclotron periods. Even at a wave frequency of 1/10 of the cyclotron frequency a Poincaré plot is quite stochastic for $k^2\Phi_0=1$. Note that this is a collisionless result.

Figure 2 shows the variation of the extent of the heating domain in $k\rho$ versus wave frequency for $k^2\Phi_0=0.36$, 0.8, and 2.6. For a small wave amplitude some peaking can indeed be seen at low-order (small) integer fractions, as predicted by the Mathieu equation approximation. As the amplitude increases, however, nonlinear generation of many fixed points produces chaos which smoothes out the resonance structures and makes the extent of the domain almost linear in ω. Of course in the limit of $\omega\rightarrow 0$ the motion is not stochastic, and there is no real heating, only large amplitude excursions in the potential. For the two larger amplitude plots an X indicates the frequency for the onset of chaos. For

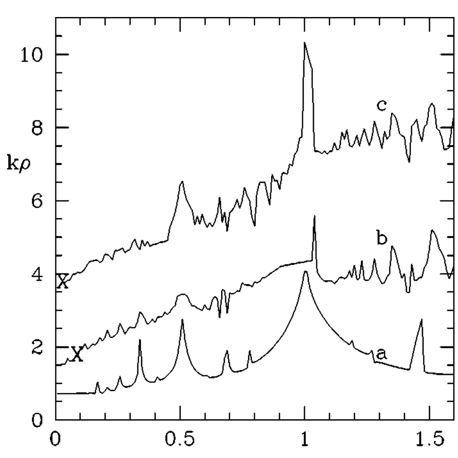

FIG. 2. Heating domain vs ω for $k^2\Phi_0$ = (a) 0.36, (b) 0.8, (c) 2.6.

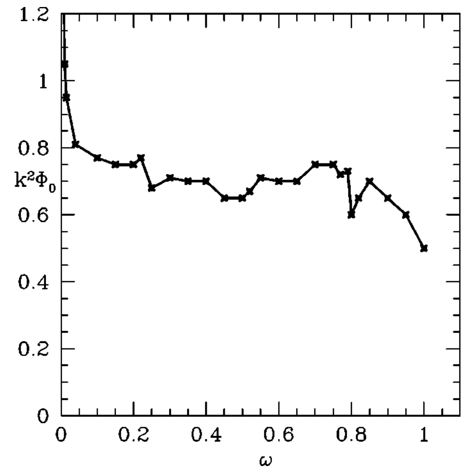

FIG. 3. Stochastic threshold in the ω, $k^2\Phi_0$ plane.

$k^2\Phi_0=0.36$, curve a, there is no chaos, only large scale convective motion, even at $\omega=1$. The onset of chaos at large wave amplitude as a function of ω is shown in Fig. 3. This plot was obtained by examination of the Poincaré plot produced by advancing a distribution with initially $k\rho<0.01$. For values of $k^2\Phi_0$ above the line there is a significant stochastic domain extending from $k\rho=0$ to an upper bound increasing with ω, as seen in Fig. 2. It is interesting to speculate the behaviors in the $|\mathbf{k}\cdot\mathbf{x}_g|\gg 1$ limit. Since this can be alternatively viewed as the $\omega,\Omega_c\rightarrow 0^+$ limit, particles will be trapped by the large-amplitude electrostatic wave for a long time, $t\sim\sqrt{k^2\Phi_0}/(\omega\Omega_c)$, and consequently, experience less efficient heating. It thus may be reasonable to conjecture that the most efficient heating occurs at intermediate wave amplitudes, $k^2\Phi_0\sim 1$. This, however, remains to be verified.

Alfvén waves, either excited spontaneously or by external sources, have been observed or predicted to be present in plasmas with parameters ranging from those of laboratory to space and astrophysical environments. Interaction between Alfvén waves and charged particles thus plays a crucial role in many plasma dynamical processes. Previous theoretical investigations of heating mechanisms have nearly always been based on the existence of the primary cyclotron resonance, i.e., $\omega-k_zu_z\pm n\Omega_c\simeq 0$ where ω and $\mathbf{k}$ are, respectively, the angular frequency and wave vector of the Alfvén wave, $\mathbf{B}_0=B_0\hat{z}$ is the confining magnetic field, $\mathbf{u}$ is the laboratory frame particle velocity, $n\geq 1$ and $\pm$ corresponds to right $(+)$ and left $(-)$ circular polarization. Such resonances will change the magnetic moment $\mu=u_\perp^2/2B_0$ leading to pitch angle scattering and heating. (The $n=0$ resonance relies on the compressional component of the wave magnetic field, which is generally negligible, only energizes particles along $\mathbf{B}_0$, and is not of interest here.) Here $\mathbf{u}_\perp$ is the velocity perpendicular to $\mathbf{B}_0$. Since for the Alfvén waves $\omega\simeq k_zv_A$ with $v_A=B/(4\pi n_0m_i)^{1/2}$ the Alfvén velocity, the cyclotron resonance condition becomes $k_zv_z=k_z(u_z-v_A)\simeq\pm n\Omega_c$, where $\mathbf{v}$ is the particle velocity in the wave frame. Noting that typically $|k_zv_A|<|n\Omega_c|$, the resonance condition generally requires that u_z be super Alfvénic, a condition often not satisfied. Wu and co-workers[8] have examined nonlinear interactions between ions and Alfvén waves under nonresonant conditions using a one dimensional ($\mathbf{k}=k_z\hat{z}$) model. Their results indicate that while the Alfvén waves can lead to large amplitude oscillations in the ion motion, there is no stochastic heating. That finite $\mathbf{k}_\perp$ is necessary for stochastic heating has been noted earlier,[9] but only for cases in which the cyclotron resonance condition was satisfied.

For a sufficiently large-amplitude, obliquely propagating ($\mathbf{k}=k_z\hat{z}+\mathbf{k}_\perp$) wave, there indeed exists efficient stochastic ion pitch angle scattering and heating by the Alfvén wave even when $k_z v_z$ is only a small fraction of Ω_c. Note, for cold ions in the laboratory frame, $v_z=-v_A$ so $\omega=-k_z v_z$ and this condition becomes $\omega \ll \Omega_c$.

The physics of this stochastic heating is qualitatively similar to that due to a perpendicularly propagating electrostatic wave with a frequency a small fraction of Ω_c, discussed above. To demonstrate this similarity, consider a linearly polarized Alfvén wave in the laboratory frame X, Y, Z, given by $\mathbf{B}_w=B_w\hat{y}\cos(\psi)$ with $\psi=\mathbf{k}\cdot\mathbf{X}-\omega t$. Furthermore, let us consider the ions initially cold in the laboratory frame, so that $\omega=-k_z v_z=k_z v_A$ at $t=0$. Extension to finite $\mathbf{u}$ at $t=0$ is straightforward. Again, take the units of time to be given by Ω_c, and normalize the field to B_0. In the wave frame $\mathbf{x}=\mathbf{X}-v_A t\hat{z}$, we have $\psi=k_x x+k_z z$ and the wave electric field is removed. The equation of motion then becomes $\dot{\mathbf{v}}=\mathbf{v}\times(\mathbf{B}_0+\mathbf{B}_w)$ and the velocity $v=v_A$ is constant in time in this frame. The wave electric field is removed in this frame and the equation of motion then become $\dot{\mathbf{v}}=\mathbf{v}\times(\mathbf{B}_0+\mathbf{B}_w)$. Dimensionless numbers characterizing the problem are then $k_x v$, $k_z v=\omega/\Omega_c$, and the wave magnitude B_w/B_0.

Specifically, the equations of motion become $\dot{v}_x=v_y-v_z B_w\cos\psi$, $v_y=x_0-x$, $\dot{v}_z=v_x B_w\cos\psi$, giving $d^2x/dt^2+x=x_0-v_z B_w\cos\psi$ and the instantaneous location of a particle in phase space is given by x,z and the pitch $\lambda=v_z/v$.

To first order in B_w we have $d^2x/dt^2+x=x_0-v_z(0)B_w\cos\psi$, equivalent to Eq. (1), with $\omega/\Omega_c=k_z v_z(0)/\Omega_c$ playing the role of the frequency of the electrostatic wave, and $k_x v_z(0)B_w/(B_0\Omega_c)$ the nonlinearity parameter, so we again find a driven Mathieu equation with unstable solutions for $\omega \simeq 2/q$ with q integer. Thus, there are resonances at many values of particle pitch in the wave frame. However, note that $k_x=0$ implies no nonlinear interaction. Physically, $k_x v_z B_w/B_0\Omega_c$ corresponds to, again, $|\mathbf{k}\cdot\mathbf{x}_g|$ with $\mathbf{x}_g$ being the wave-induced guiding-center curvature-drift displacement.

Thus, for small B_w, we expect the same resonances as found in the electrostatic wave analysis. For larger values of B_w, however, the approximation of $v_z=v_z(0)$ will be invalid in the differential equation for x and the motion will be more complex.

To study the resonances, again take a Poincaré section of λ,ψ, formed by taking points when $v_y=0$ and $\dot{v}_y>0$ for a distribution at a fixed energy in the wave frame. At large wave amplitude these resonances produce stochastic motion, and hence allow nonadiabatic, permanent change in pitch λ. Transformed back to the laboratory frame, such a change in pitch is equivalent to particle heating as well as pitch angle scattering.

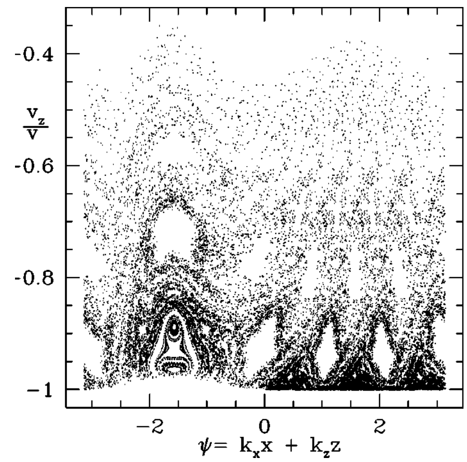

FIG. 4. Poincaré, $B_w=0.25$. $k_x v=0.27$, $\omega=0.25$.

Noting that waves of left-hand polarization are often excited in space plasmas,[10] we shall consider in the following only a left-hand polarized component, resonating preferably with particles having $v_z<0$. Thus, we have, again in the wave frame, $\mathbf{B}_w=-B_w\hat{x}\cos(\alpha)\sin(\psi)+B_w\hat{y}\cos\psi+B_w\hat{z}\sin(\alpha)\sin(\psi)$ with $\psi=k_x x+k_z z$ and $\tan(\alpha)=k_x/k_z$. In the laboratory frame, the wave propagates in the positive z direction, and, in the wave frame, $v_z/v=-1$ for an initially cold ion distribution.

Figure 4 shows a Poincaré plot for a left-hand circularly polarized wave with $B_w=0.25$, $k_x v=0.27$, $\omega=0.25$. All particles were initiated with $v_z/v<-0.99$, i.e., the initial ion distribution in the lab frame was cold. At this amplitude the lower part of the plot is chaotic, and ions can readily diffuse from $v_z/v=-1$ to values near -0.4. Transforming back to the rest frame, the final ion distribution has a spread in perpendicular velocity much larger than the spread in parallel velocity.

In Fig. 5 is shown a numerical determination of the stochastic threshold in the plane of $k_x v$, B_w for frequencies of $\omega=0.1,0.25,0.5$. Above the line there is a significant chaotic domain leading to heating. Note that stochastic threshold is achieved when the nonlinearity parameter $k_x v_z(0)B_w/(B_0\Omega_c)$ is of order 0.1, i.e., at a much lower level than is the case for an electrostatic wave. This, as

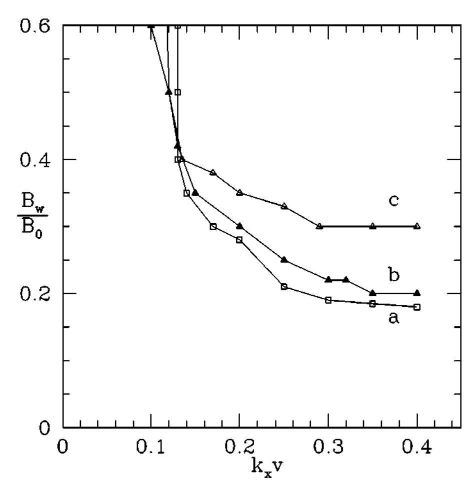

FIG. 5. Stochastic threshold, (a) $\omega=0.5$, (b) $\omega=0.25$, (c) $\omega=0.1$.

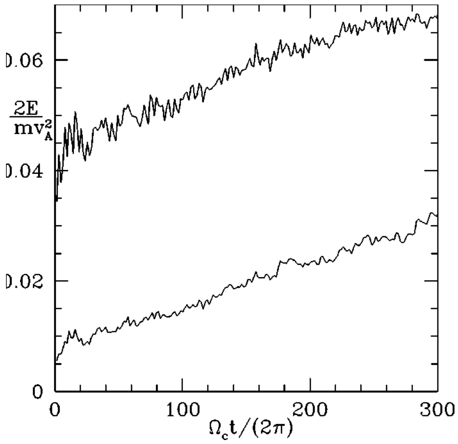

FIG. 6. $E_\perp$ (larger) and $E_\parallel$ vs t, $B_w = 0.25$, $k_x v = 0.27$, $\omega = 0.25$.

shown in the linearly polarized case, is due to the existence of resonances at many values of the pitch angle. Again, the issue whether stochastic heating exists for $B_w \gg B_0$ remains to be studied.

In Fig. 6 is shown the heating of an initially cold distribution. Since the distribution begins with $\lambda = -1$, the ions mainly gain perpendicular energy and thus we see $E_\perp > E_\parallel$ with $E_\perp$ and $E_\parallel$ being, respectively, the energies perpendicular and parallel to $\mathbf{B}_0$. Two hundred cyclotron periods is sufficient to heat to $v \simeq 0.25 v_A$ for these parameters. These results may provide an interesting new mechanism of solar corona heating by Alfvén waves.[11] The ion distribution produced in this figure has a perpendicular thermal velocity of 250 km/s taking $v_A \sim 10^3$ km/s in the lower solar corona. Furthermore, since energization increases with ω/Ω_c, this heating mechanism will preferentially energize heavier mass (lower Ω_c) ions. These features are consistent with observations.[12]

In conclusion, we have demonstrated that, given sufficiently large amplitudes, significant perpendicular heating can be obtained at a fraction of the cyclotron frequency both for the case of a longitudinal wave propagating across a constant magnetic field and for large amplitude obliquely propagating Alfvén waves. The existence of nonlinear resonances at fractions of the cyclotron frequency is, we believe, a generic phenomenon and may be expected to occur for other types of waves.

ACKNOWLEDGMENTS

L.C. acknowledges useful discussions with Y. Lin and C.S. Wu.

This work was supported by the U.S. Department of Energy Grant No. DE-FG03-94ER54271 and under Contract Number DE-AC02-76-CHO3073 and NSF No. Grant ATM-9971529.

[1] M. A. Lieberman and A. J. Lichtenberg, *Regular and Chaotic Dynamics* (Springer Verlag, New York, 1983), pp. 339–345.
[2] A. K. Nekrasov, Nucl. Fusion **10**, 387 (1970).
[3] F. Jaeger and A. J. Lichtenberg, Ann. Phys. **71**, 319 (1972).
[4] M. A. Lieberman and A. J. Lichtenberg Plasma Phys. **15**, 125 (1973).
[5] C. F. F. Karney, Phys. Fluids **22**, 2188 (1979).
[6] H. Abe, H. Okada, R. Itatani, M. Ono, and H. Okuda, Phys. Rev. Lett. **53**, 1153 (1984).
[7] H. Abe, H. Momota, R. Itatani, and A. Fukuyama, Phys. Fluids **23**, 2417 (1980).
[8] C. S. Wu, P. H. Yoon, and J. K. Chao, Phys. Plasmas **4**, 856 (1997).
[9] Y. Lin and L. C. Lee, Space Sci. Rev. **65**, 59 (1994).
[10] B. J. Anderson, R. E. Erlandson, and L. J. Zanetti, J. Geophys. Res. **97**, 3089 (1992).
[11] See, for example, the review by U. Narian and P. Ulmschneider, Space Sci. Rev. **75**, 453 (1996).
[12] S. R. Cranmer *et al.*, Astrophys. J. **511**, 481 (1999); S. R. Cranmer, G. B. Field, and J. L. Kohl, *ibid.* **518**, 937 (1999), and references therein.