Non-linear zonal dynamics of drift and drift-Alfvén turbulence in tokamak plasmas

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Abstract. The present work addresses the issue of identifying the major non-linear physics processes which may regulate drift and drift-Alfvén turbulence using a weak turbulence approach. Within this framework, on the basis of the non-linear gyrokinetic equation for both electrons and ions, an analytic theory is presented for non-linear zonal dynamics described in terms of two axisymmetric potentials, $\delta\phi_z$ and $\delta A_{\parallel z}$, which spatially depend only on a (magnetic) flux co-ordinate. Spontaneous excitation of zonal flows by electrostatic drift microinstabilities is demonstrated both analytically and by direct 3-D gyrokinetic simulations. Direct comparisons indicate good agreement between analytic expressions of the zonal flow growth rate and numerical simulation results for ion temperature gradient driven modes. Analogously, it is shown that zonal flows may be spontaneously excited by drift-Alfvén turbulence, in the form of modulational instability of the radial envelope of the mode as well, whereas in general, excitations of zonal currents are possible but have little feedback on the turbulence itself.

1. Introduction

In recent years, there has been increasing attention devoted to exploring the non-linear dynamics of zonal flow [1] associated with electrostatic drift type turbulence [2–4]. On the other hand, despite it being well known how electrostatic drift modes couple to the electromagnetic shear Alfvén wave as the plasma β (or $R_0\beta'$) increases [5–7], little effort has been devoted so far to investigating the non-linear zonal dynamics of drift-Alfvén turbulence.

The present work addresses the issue of identifying the major non-linear physics processes which may regulate drift and drift-Alfvén turbulence using a weak turbulence approach. Within this framework, on the basis of the non-linear gyrokinetic equation [8] for both electrons and ions, we present an analytic theory for non-linear zonal dynamics described in terms of two axisymmetric potentials, $\delta\phi_z$ and $\delta A_{\parallel z}$, which spatially depend only on a (magnetic) flux co-ordinate. Physically $\delta\phi_z$ is associated with zonal flow formation, while $\delta A_{\parallel z}$ corresponds to zonal currents $\delta j_{\parallel z} = -(c/4\pi)\nabla_\perp^2 \delta A_{\parallel z}$. The introduction of a zonal vector potential, $\delta A_{\parallel z}$, is one of the characteristic differences of the electromagnetic with respect to the electrostatic case.

Zonal potentials are characterized by time variations on typical scales which are long compared with those characteristic of the drift-Alfvén instabilities. This specific ordering of timescales, which formally requires proximity to the marginal stability such that the linear growth rate is smaller than the mode frequency, will be exploited for explicitly manipulating formal expressions in the theoretical analysis. In contrast to other approaches, however, which also assume slow radial variations of the zonal fields (k_z^{-1}) with respect to the typical spatial scale of the background turbulence (k_r^{-1}), we generally take $k_z \approx k_r$, although we still assume $|\partial_r k_z/k_z^2| \ll 1$ for consistency of our eikonal approach. In this respect our work is the generalization of Ref. [9], which demonstrated that zonal flows can be spontaneously excited by electrostatic drift turbulence and that these are characterized by $k_z \approx k_r$ (Fig. 1). In the present work, we show that zonal flows in toroidal equilibria can be spontaneously excited via modulations of the radial structure (envelope) of a single-n coherent drift wave, with n the toroidal mode number. In this framework, the turbulent state and the non-linear couplings among different n's will be manifest only via zonal dynamics. Similarly to Ref. [9], the present theory is strictly applicable to toroidal

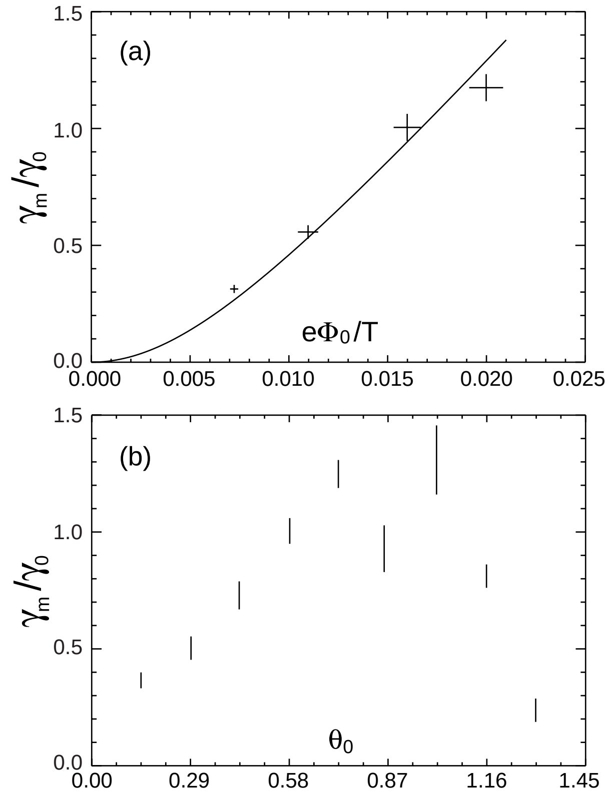

Figure 1. (a) Analytic prediction of zonal flow growth rate (normalized to the linear growth rate γ_0) versus mode amplitude (solid line) compared with gyrokinetic simulation results. (b) Zonal flow growth rate versus $\theta_0 = k_z/nq'$, for fixed mode amplitude. From Ref. [9].

plasma equilibria, where poloidal asymmetry forces each mode to be (at least in the linear limit) the superposition of many poloidal harmonics m, characterized by the same n. In this respect, the present theoretical analysis is a systematic treatment of the radial mode structure (envelope) of zonal fields and drift turbulence in the general electromagnetic case, including slow time evolutions and accounting for linear (toroidal) and non-linear mode couplings on the same footing. More specifically, we demonstrate that zonal flows ($\delta\phi_z$) are due to charge separation effects associated with both finite ion Larmor radius and finite ion orbit width effects (magnetic curvature), whereas zonal currents ($\delta A_{\|z}$) are due to parallel electron pressure imbalance (cf. Ref. [10]).

Spontaneous excitation of zonal flows by electrostatic drift microinstabilities is demonstrated both analytically and by direct 3-D gyrokinetic simulations [9]. Direct comparisons indicate good agreement between analytic expressions of the zonal flow growth rate and numerical simulation results for ion temperature gradient (ITG) modes. Analogously, we show that zonal flows may be spontaneously excited by drift-Alfvén turbulence, in the form of modulational instability of the radial envelope of the mode as well. From the analytic expression for the growth rate of the spontaneously excited zonal flows ($\delta\phi_z$) we show how no flow generation is expected for a pure shear Alfvén wave, owing to the peculiar nature of the Alfvénic state. Meanwhile we also demonstrate that, in general, zonal currents are also excited but they have negligible effect on the turbulence itself. The general results obtained within this theoretical model are also applied to Alfvénic oscillations, such as the kinetic Alfvén waves and the more recently discussed Alfvén ITG (AITG) [6] mode.

2. Theoretical model

Here we strictly follow Ref. [9] and assume a low β ($\beta = 8\pi p/B^2$) toroidal equilibrium with major radius R_0 and minor radius a, with typically $R_0/a = 1/\epsilon \gg 1$. For simplicity we also take the case of shifted circular magnetic flux surfaces. In this case we can describe drift wave dynamics in terms of two scalar fields: the scalar potential $\delta\phi$ and the parallel vector potential $\delta A_\|$ fluctuations. For both fluctuating fields, as stated in Section 1, we describe the non-linear dynamic evolution in terms of a four mode coupling scheme, i.e. each electromagnetic fluctuation is taken to be coherent and composed of a single $n \neq 0$ drift wave ($\delta\phi_d, \delta A_{\|d}$) and a zonal perturbation ($\delta\phi_z, \delta A_{\|z}$). For example, for scalar potential fluctuations we take

$$\delta\phi_d = \delta\phi_0 + \delta\phi_+ + \delta\phi_- \tag{1a}$$

$$\delta\phi_0 = e^{i\int n\theta_k dq + in\varphi} \sum_m e^{-im\vartheta}\phi_0(nq - m) + \text{c.c.} \tag{1b}$$

$$\delta\phi_\pm = \begin{pmatrix} e^{i\int n\theta_k dq} \\ e^{-i\int n\theta_k^* dq} \end{pmatrix} e^{\pm in\varphi + i\int k_z dr} \times \sum_m e^{\mp im\vartheta}\phi_\pm(nq - m) + \text{c.c.} \tag{1c}$$

$$\delta\phi_z = e^{i\int k_z dr}\phi_z + \text{c.c.} \tag{1d}$$

where (r, φ, ϑ) are toroidal co-ordinates, and an analogue decomposition is assumed for fluctuating parallel vector potentials. Here θ_k is the eikonal describing the radial structure of the drift wave radial envelope and q is the safety factor. Thus Eqs (1) suggest that zonal fields may actually be considered as radial modulations of the drift wave envelope, while the (±) modes are simply upper and lower sidebands due to zonal field modulations of the drift wave [9]. Furthermore, we have adopted the convention that, in the expressions involving the ± sidebands, the first row in a two component array will refer to the + while the second row will refer to the − sideband. The same notation will be used throughout.

We first derive non-linear equations for zonal fields from the quasi-neutrality condition and parallel Ampère's law. Here we only report the final results of such derivations in the small ion Larmor radius (ρ_{Li}) limit; details will be given elsewhere. Contrary to the electrostatic limit, where the electron response to an $n \neq 0$ perturbation is adiabatic, and thus only ions contribute to the non-linear dynamics, electron non-linearities are important in the general electromagnetic case. Assuming $k_\perp^2 \rho_{Li}^2 \ll 1$, the non-linear coupling coefficients are formally of the Hasegawa–Mima type and the quasi-neutrality condition reads

$$\partial_t \chi_{iz} \delta\phi_z = \frac{c}{B} k_\vartheta k_z k_z^2 \rho_{Li}^2 \left[\left(\alpha_0 - \left| \frac{k_\parallel v_A}{\omega_0} \right|^2 \right) \langle\langle |\Psi_0|^2 \rangle\rangle + 2\alpha_0 \,\mathrm{Re} \langle\langle (\Phi_0 - \Psi_0)^* \Psi_0 \rangle\rangle + \alpha_0 \langle\langle |\Phi_0 - \Psi_0|^2 \rangle\rangle \right] (A_0^* A_+ - A_0 A_-). \quad (2)$$

Here we have assumed $\delta\phi_0 \approx \exp(-i\omega_0 t)$, $k_\vartheta \equiv m/r$, $\rho_{Li}^2 = (T_i/m_i)/\omega_{ci}^2$, ω_{ci} is the ion cyclotron frequency and $v_A \equiv B/\sqrt{4\pi n m_i}$ is the Alfvén speed. We also have assumed one ion species only, with unit electric charge e and density n, and introduced the notations $\chi_{iz} \simeq 1.6 q^2 \epsilon^{-1/2} k_z^2 \rho_{Li}^2$ [11], $\alpha_0 \equiv 1 + \delta P_{\perp i0}/(ne\delta\phi_0)$, $\boldsymbol{b} \cdot \boldsymbol{\nabla} \delta\psi \equiv -(1/c)\partial_t \delta A_\parallel$, Φ_0 indicates the symmetric Fourier transform into ballooning space of ϕ_0 in Eq. (1) and similar notations are used for the other scalar fields. Furthermore, we have omitted for simplicity collisional damping of $\delta\phi_z$ [12], $\langle\langle \ldots \rangle\rangle$ stands for integration over ballooning space, $|k_\parallel|^2 \langle\langle |\Psi_0|^2 \rangle\rangle \equiv \langle\langle |\partial_\theta \Psi_0 / q R_0|^2 \rangle\rangle$, θ is the 'angle-like' co-ordinate in ballooning space, and A_0 and $A_\pm$ indicate the amplitude of radial envelopes of the drift wave and sidebands at the current radial position. Similarly, from the parallel Ampère's law we obtain

$$\partial_t \delta A_{\parallel z} = \frac{c}{B} k_\vartheta k_z k_z^2 \delta_e^2 \left\langle\!\left\langle \mathrm{Re}\left(\frac{k_\parallel c}{\omega_0} (\Phi_0 - \Psi_0)^* \Psi_0 \right) \right\rangle\!\right\rangle \times (A_0^* A_+ - A_0 A_-). \quad (3)$$

Here the presence of $\delta_e^2 = c^2/\omega_{pe}^2$ is a consequence of the strong shielding effect of parallel electron current on the electron collisionless skin depth. Furthermore, the forced response in $\delta A_{\parallel z}$ has been neglected since it is of order $(\omega_z/\omega_0)(k_z^2\delta_e^2)^{-1}(k_\perp^2\rho_{Li}^2)^{-1}$ with respect to the spontaneously excited component of Eq. (3). A direct comparison of Eqs (2) and (3) indicates that both zonal fields may be spontaneously excited except for a pure shear Alfvén wave, for which $\omega_0^2 = k_\parallel^2 v_A^2$, $\alpha_0 = 1$ and $\Phi_0 = \Psi_0$. In general, however, zonal flows can be efficiently excited via $\delta\phi_z$, whereas zonal currents (or poloidal magnetic fields) are strongly reduced because of electron shielding on scale lengths larger that δ_e. We also note that, typically,

$$\frac{\omega_0}{k_\parallel c} \delta A_{\parallel z} \approx k_z^2 \delta_e^2 \delta\phi_z \ll \delta\phi_z$$

which will make it possible to neglect the effect of $\delta A_{\parallel z}$ below.

The drift wave non-linear equations are the quasi-neutrality condition

$$\frac{ne^2}{T_i}\left(1 + \frac{T_i}{T_e}\right)\delta\phi_k = \left\langle eJ_0(\gamma)\overline{\delta H}_i \right\rangle_k - \left\langle e\overline{\delta H}_e \right\rangle_k \quad (4)$$

and the vorticity equation

$$\begin{aligned} &B\,\partial_\ell\left(k_\perp^2 \frac{\partial_\ell \delta\psi_k}{B}\right) + \frac{\omega^2}{v_A^2}\frac{k_\perp^2}{b_i}\left[\left(1 - \frac{\omega_{*ni}}{\omega}\right)[1-\Gamma_0(b_i)] - \frac{\omega_{*Ti}}{\omega} b_i [\Gamma_0(b_i) - \Gamma_1(b_i)]\right]\delta\phi_k \\ &= \frac{4\pi}{c^2}\sum_{e,i}\langle e\omega\omega_d J_0 \overline{\delta H}\rangle_k \\ &+ \frac{\boldsymbol{b}\cdot(\boldsymbol{k}''_\perp \times \boldsymbol{k}'_\perp)}{cB}\partial_t(\delta A_{\parallel,k'}\nabla_\perp^2 \delta A_{\parallel k''})_k \\ &+ \frac{4\pi}{c^2}\partial_t\left\langle e\frac{c}{B}\boldsymbol{b}\cdot(\boldsymbol{k}''_\perp \times \boldsymbol{k}'_\perp) \times [J_0(\gamma)J_0(\gamma') - J_0(\gamma'')]\delta L_{k'}\overline{\delta H}_{ik''}\right\rangle_k . \end{aligned} \quad (5)$$

In Eqs (4) and (5), the subscript k stands for 0 or ± depending on whether the drift wave or its sidebands are considered, simple angular brackets $\langle \ldots \rangle$ denote velocity space integration, $\gamma \equiv k_\perp v_\perp/\omega_{ci}$, J_0 is the Bessel function of zero order, $\partial_\ell \equiv \boldsymbol{b}\cdot\boldsymbol{\nabla}$, $b_i = k_\perp^2\rho_{Li}^2$, $\Gamma_{0,1}(b_i) \equiv I_{0,1}(b_i)\exp(-b_i)$, ω_{*ni} and ω_{*Ti} are the ion diamagnetic frequencies associated with density and temperature gradients, respectively, ω_d is the magnetic drift frequency, $\boldsymbol{k} = \boldsymbol{k}' + \boldsymbol{k}''$, $\delta L_k \equiv \delta\phi_k - (v_\parallel/c)\delta A_{\parallel k}$ and the fluctuating particle distribution functions have been decomposed in adiabatic and non-adiabatic responses as

$$\delta F = \frac{e}{m}\delta\phi\frac{\partial}{\partial v^2/2}F_0 + \sum_{\boldsymbol{k}_\perp}\exp(-i\boldsymbol{k}_\perp\cdot\boldsymbol{v}\times\boldsymbol{b}/\omega_c)\overline{\delta H}_k. \quad (6)$$

The non-adiabatic response of the particle distribution function, $\overline{\delta H}$, is obtained from the non-linear gyrokinetic equation [8]:

$$\left(\partial_t + v_\parallel \partial_\ell + i\omega_d\right)_k \overline{\delta H}_k = i\frac{e}{m} Q F_0 J_0(\gamma)\delta L_k - \frac{c}{B}\boldsymbol{b}\cdot\left(\boldsymbol{k}''_\perp \times \boldsymbol{k}'_\perp\right) J_0(\gamma')\delta L_{k'}\overline{\delta H}_{k''} \tag{7a}$$

$$QF_0 = \omega_k \frac{\partial F_0}{\partial v^2/2} + \boldsymbol{k}\cdot\frac{\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}}{\omega_c}F_0. \tag{7b}$$

In Eqs (7), the linear response $\propto QF_0$ and the 'generalized' $\boldsymbol{E}\times\boldsymbol{B}$ non-linearity (in the guiding centre moving frame $\delta\phi \to \delta\phi - (v_\parallel/c)\delta A_\parallel$) are readily recognized.

Equations (4) and (5) are further simplified when we decompose the linear particle response to the fluctuating fields as [13]

$$\overline{\delta H}^{LIN} = -\frac{e}{m}J_0(\gamma)\frac{QF_0}{\omega}\delta\psi + \delta K \tag{8}$$

where the linearized gyrokinetic equation for δK is readily derived from Eq. (7a) and may be found in Ref. [13]. It is then readily shown that the quasi-neutrality condition, Eq. (4), can be cast into the form

$$\frac{ne^2}{T_i}\left\{\left(1+\frac{T_i}{T_e}\right)(\delta\phi-\delta\psi)_k + \left[\left(1-\frac{\omega_{*ni}}{\omega}\right)[1-\Gamma_0(b_i)] - \frac{\omega_{*Ti}}{\omega}b_i[\Gamma_0(b_i)-\Gamma_1(b_i)]\right]\right\}\delta\psi_k - \sum_{e,i}\langle eJ_0(\gamma)\delta K\rangle_k = -\frac{i}{\omega_k}\left\langle e\frac{c}{B}\boldsymbol{b}\cdot(\boldsymbol{k}''_\perp\times\boldsymbol{k}'_\perp) \times[J_0(\gamma)J_0(\gamma')-J_0(\gamma'')]\delta L_{k'}\overline{\delta H}_{ik''}\right\rangle_k - \frac{i}{\omega_k}\left\langle e\frac{c}{B}\boldsymbol{b}\cdot(\boldsymbol{k}''_\perp\times\boldsymbol{k}'_\perp)\delta\phi_{k'}\overline{\delta H}_{ek''}\right\rangle_k - \langle e\overline{\delta H}_e^{NL}\rangle_k \tag{9}$$

where $\overline{\delta H}_e^{NL}$ indicates the non-linear non-adiabatic electron response only, which vanishes in the electrostatic limit, as stated above. In Eq. (9), the non-linear ion response has been computed assuming $|k_\parallel v_{thi}|, |\omega_{di}| \ll |\omega_0|$, $v_{thi} = \sqrt{T_i/m_i}$ being the ion thermal speed.

Assuming now $k_\perp^2\rho_{Li}^2 \ll 1$, consistently with Eqs (2) and (3), and introducing the notation

$$\delta K = \widehat{\delta K}_\phi(\delta\phi - \delta\psi) + \widehat{\delta K}_\psi\delta\psi \tag{10}$$

Eqs (5) and (9) for the sidebands in the ballooning space can be rewritten as

$$\left(1+\frac{T_i}{T_e} - \sum_{e,i}\langle eJ_0(\gamma)\widehat{\delta K}_\phi\rangle_\pm\right)A_\pm\begin{pmatrix}\Phi_0-\Psi_0\\ \Phi_0^*-\Psi_0^*\end{pmatrix} + \left[\left(1-\frac{\omega_{*pi}}{\omega}\right)b_{i\pm} - \sum_{e,i}\langle eJ_0(\gamma)\widehat{\delta K}_\psi\rangle_\pm\right]A_\pm\begin{pmatrix}\Psi_0\\ \Psi_0^*\end{pmatrix} = -\frac{i}{\omega_0}\frac{c}{B}\frac{T_i}{T_e}k_\vartheta k_z\delta\phi_z\left[\left(1+\frac{\omega_{*ni}}{\omega_0}\frac{T_e}{T_i}\right)\begin{pmatrix}A_0\Psi_0\\ A_0^*\Psi_0^*\end{pmatrix} - \begin{pmatrix}A_0(\Phi_0-\Psi_0)\\ A_0^*(\Phi_0^*-\Psi_0^*)\end{pmatrix}\right]. \tag{11}$$

$$\left\{\partial_\theta\left(\frac{k_\perp^2}{k_\vartheta^2}\partial_\theta\right) + \frac{\omega^2}{\omega_A^2}\frac{k_\perp^2}{k_\vartheta^2}\left[\left(1-\frac{\omega_{*pi}}{\omega}\right) - \frac{3}{4}b_i\left(1-\frac{\omega_{*pi}}{\omega}-\frac{\omega_{*Ti}}{\omega}\right)\right] - \frac{4\pi q^2R_0^2}{k_\vartheta^2c^2}\sum_{e,i}\langle e\omega\omega_dJ_0\widehat{\delta K}_\psi\rangle\right\}_\pm A_\pm\begin{pmatrix}\Psi_0\\ \Psi_0^*\end{pmatrix} + \left\{\frac{\omega^2}{\omega_A^2}\frac{k_\perp^2}{k_\vartheta^2}\left[\left(1-\frac{\omega_{*pi}}{\omega}\right) - \frac{3}{4}b_i\left(1-\frac{\omega_{*pi}}{\omega}-\frac{\omega_{*Ti}}{\omega}\right)\right] - \frac{4\pi q^2R_0^2}{k_\vartheta^2c^2}\sum_{e,i}\langle e\omega\omega_dJ_0\widehat{\delta K}_\phi\rangle\right\}_\pm A_\pm\begin{pmatrix}\Phi_0-\Psi_0\\ \Phi_0^*-\Psi_0^*\end{pmatrix} = \frac{4\pi i\omega_0}{k_\vartheta^2c^2}\frac{c}{B}\frac{ne^2}{T_i}q^2R_0^2k_\vartheta k_z\delta\phi_zb_i\begin{pmatrix}A_0\Phi_0\\ A_0^*\Phi_0^*\end{pmatrix}. \tag{12}$$

Equations (2), (11) and (12), together with Eq. (7a), are the basis for our analytic investigations described in the next section.

3. Some applications

In the electrostatic limit [9], $\Psi_0 \to 0$, we obtain from Eq. (11)

$$D_{S\pm}A_\pm = \frac{i}{\omega_0}\frac{c}{B}\frac{T_i}{T_e}k_\vartheta k_z\delta\phi_z\begin{pmatrix}A_0\\ A_0^*\end{pmatrix} \tag{13}$$

where

$$D_{S\pm} = \left\langle\left\langle\left(1+\frac{T_i}{T_e}-\sum_{e,i}\langle eJ_0(\gamma)\widehat{\delta K}_\phi\rangle_\pm\right)\times\begin{pmatrix}\Phi_0^2\\ \Phi_0^{*2}\end{pmatrix}\right\rangle\right\rangle\left\langle\left\langle\times\begin{pmatrix}\Phi_0^2\\ \Phi_0^{*2}\end{pmatrix}\right\rangle\right\rangle^{-1} \tag{14}$$

and [9] $D_{S\pm} \simeq i(\partial D_{S0r}/\partial\omega_0)(-i\Delta \pm \Gamma_z \pm \gamma_d)$, $\Delta = (k_z^2/2)(\partial^2 D_{S0r}/\partial k_r^2)/(\partial D_{S0r}/\partial\omega_0)$ is the frequency mismatch, $k_r = nq'\theta_k$, $\Gamma_z = -i\omega_z$ and γ_d is the sideband damping [9]. Substituting Eq. (13) into Eq. (2), we readily obtain a non-linear dispersion relation for Γ_z, which, in the $|\Delta| \ll \gamma_d, \gamma_M$ limit, reads

$$\Gamma_z = -\gamma_d/2 + \left(\gamma_M^2 + \gamma_d^2/4\right)^{1/2} \tag{15}$$

where $\gamma_M^2 = (2\alpha_0\epsilon^{1/2}/1.6q^2)(T_i/T_e)(\omega_0\partial D_{S0r}/\partial\omega_0)^{-1} \times k_z^2\rho_{Li}^2 k_\vartheta^2 v_{thi}^2 \langle\langle|eA_0\Phi_0/T_i|^2\rangle\rangle$. Including finite zonal flow collisional damping into Eq. (2), $\nu_z \simeq (1.5\epsilon\tau_{ii})^{-1}$ [12], would have produced a threshold condition $\gamma_M^2 \geq \nu_z\gamma_d$ on the modulational instability growth rate Γ_z of Eq. (15). In Fig. 1, Γ_z as obtained from Eq. (15) is shown to be in good agreement with the results obtained by direct 3-D gyrokinetic simulations [9] of ITG modes, in which $\gamma_d \simeq 1.5\gamma_0$, γ_0 being the linear growth rate of the mode. Non-linear equations for mode amplitudes have been recently derived [9] and they demonstrate saturation of the linearly unstable modes via coupling to the stable envelope sidebands and oscillatory behaviours in the drift wave intensity and zonal flows [9].

For electromagnetic modes, and more specifically for Alfvénic type waves, we typically have $|\Phi_0 - \Psi_0| \ll |\Psi_0|$ in Eqs (11) and (12). In fact, assuming $k_\parallel^2 q^2 R_0^2 \ll 1$ [6], we have from Eq. (11)

$$\begin{pmatrix}\Phi_0 - \Psi_0 \\ \Phi_0^* - \Psi_0^*\end{pmatrix} A_\pm \simeq -\left(\frac{(k_\parallel^2 v_A^2/\omega^2)b_i}{T_i/T_e + \omega_{*ni}/\omega}\right)_\pm \begin{pmatrix}\Psi_0 \\ \Psi_0^*\end{pmatrix} \times A_\pm - i\frac{c}{B}\frac{k_\vartheta k_z}{\omega_0}\delta\phi_z \begin{pmatrix}A_0\Psi_0 \\ A_0^*\Psi_0^*\end{pmatrix} \tag{16}$$

where we recall that $k_\parallel^2$, in the present treatment, stands for an operator in the ballooning space. Substituting back into Eq. (12), this yields [6]

$$\mathcal{L}_{M\pm}\begin{pmatrix}\Psi_0 \\ \Psi_0^*\end{pmatrix} A_\pm = i\frac{\omega_0}{\omega_A^2}\frac{c}{B}k_\vartheta k_z \frac{k_{\perp\pm}^2}{k_\vartheta^2}\delta\phi_z \times \left(1 + \frac{k_\parallel^2 v_A^2}{\omega^2}\right)_\pm \begin{pmatrix}A_0\Psi_0 \\ A_0^*\Psi_0^*\end{pmatrix} \tag{17}$$

$$\mathcal{L}_{M\pm} = \left\{\partial_\theta\left(\frac{k_\perp^2}{k_\vartheta^2}\partial_\theta\right) + \frac{\omega^2}{\omega_A^2}\frac{k_\perp^2}{k_\vartheta^2}\left[\left(1 - \frac{\omega_{*pi}}{\omega}\right)\right.\right. \times \left(1 - \frac{(k_\parallel^2 v_A^2/\omega^2)b_i}{T_i/T_e + \omega_{*ni}/\omega}\right) - \left.\frac{3}{4}b_i\left(1 - \frac{\omega_{*pi}}{\omega} - \frac{\omega_{*Ti}}{\omega}\right)\right] - \frac{4\pi q^2R_0^2}{k_\vartheta^2 c^2}\left(\sum_{e,i}\langle e\omega\omega_d J_0\widehat{\delta K}_\psi\rangle - \frac{(k_\parallel^2 v_A^2/\omega^2)b_i}{T_i/T_e + \omega_{*ni}/\omega} \times \left.\left.\sum_{e,i}\langle e\omega\omega_d J_0\widehat{\delta K}_\phi\rangle\right)\right\}_\pm . \tag{18}$$

Equation (17) can be cast into the form

$$D_{M\pm}A_\pm = i\frac{\omega_0}{\omega_A^2}\frac{c}{B}k_\vartheta k_z\delta\phi_z\left(1 + \frac{K_\parallel^2 v_A^2}{\omega^2}\right)_\pm \begin{pmatrix}A_0 \\ A_0^*\end{pmatrix} \tag{19a}$$

$$D_{M\pm} \equiv \left\langle\left\langle\begin{pmatrix}\Psi_0 \\ \Psi_0^*\end{pmatrix}\mathcal{L}_{M\pm}\begin{pmatrix}\Psi_0 \\ \Psi_0^*\end{pmatrix}\right\rangle\right\rangle \left\langle\left\langle\frac{k_{\perp\pm}^2}{k_\vartheta^2}\begin{pmatrix}\Psi_0^2 \\ \Psi_0^{2*}\end{pmatrix}\right\rangle\right\rangle^{-1} \tag{19b}$$

$$K_{\parallel\pm}^2 \equiv \left\langle\left\langle\begin{pmatrix}\Psi_0 \\ \Psi_0^*\end{pmatrix}\frac{k_{\perp\pm}^2}{k_\vartheta^2}k_{\parallel\pm}^2\begin{pmatrix}\Psi_0 \\ \Psi_0^*\end{pmatrix}\right\rangle\right\rangle \times \left\langle\left\langle\frac{k_{\perp\pm}^2}{k_\vartheta^2}\begin{pmatrix}\Psi_0^2 \\ \Psi_0^{2*}\end{pmatrix}\right\rangle\right\rangle^{-1} . \tag{19c}$$

From Eqs (19) and Eq. (2), it is possible to derive the non-linear dispersion relation for Γ_z, similar to the electrostatic case. Specifically, using $D_{M-} = D_{M+}^*$, we obtain

$$\Gamma_z = 2k_\vartheta^2\rho_{Li}^2\frac{k_z^2 v_{thi}^2}{\omega_0}\frac{\omega_0^2}{\omega_A^2}\frac{\epsilon^{1/2}}{1.6q^2}\left\langle\left\langle\left|\frac{eA_0\Psi_0}{T_i}\right|^2\right\rangle\right\rangle \times\frac{\mathrm{Im}\left[D_{M+}\left(1+K_\parallel^2 v_A^2/\omega^2\right)_-\right]}{|D_{M+}|^2}\left[\left(\alpha_0 - \left|\frac{K_\parallel^2 v_A^2}{\omega_0^2}\right|\right) - 2\alpha_0\,\mathrm{Re}\left(\frac{(K_\parallel^2 v_A^2/\omega^2)K_\perp^2\rho_{Li}^2}{T_i/T_e + \omega_{*ni}/\omega}\right)_\pm\right] \tag{20a}$$

$$K_{\parallel+}^2 K_{\perp+}^2 \equiv \left\langle\left\langle\Psi_0^* k_{\perp+}^2 k_{\parallel+}^2\Psi_0\right\rangle\right\rangle \langle\langle|\Psi_0|^2\rangle\rangle^{-1} . \tag{20b}$$

It is straightforward to further specialize Eqs (20) to the case of the kinetic Alfvén wave (KAW), for which $D_M = -q^2R_0^2K_\parallel^2 + (\omega^2/\omega_A^2)(1 - K_\perp^2\rho_{Li}^2(3/4 + T_e/T_i))$. In this case $\alpha_0 = 1$, and defining

$$\hat{\gamma}_M^2 = 2k_\vartheta^2\rho_{Li}^2 k_z^2 v_{thi}^2\frac{\epsilon^{1/2}}{1.6q^2}\left(\frac{3}{4} - \frac{T_e}{T_i}\right)K_\perp^2\rho_{Li}^2 \times\left\langle\left\langle\left|\frac{eA_0\Psi_0}{T_i}\right|^2\right\rangle\right\rangle \tag{21a}$$

$$\hat{\Delta} = \left(\frac{3}{4} + \frac{T_e}{T_i}\right)k_z^2\rho_{Li}^2\frac{\omega_0}{2} \tag{21b}$$

we obtain

$$\Gamma_{z,KAW} \simeq \hat{\gamma}_M\sqrt{1 - \hat{\Delta}^2/\hat{\gamma}_M^2} . \tag{22}$$

From Eqs (21) and (22) we see that zonal flows can be spontaneously excited by KAWs and that, as in the electrostatic case, the growth rate Γ_z above the threshold scales linearly with the wave amplitude. However, the most important feature of KAWs is that they spontaneously generate zonal flows in their propagating region for $T_e < (3/4)T_i$ and in their cut-off region for $T_e > (3/4)T_i$.

Another application of Eqs (20) is to AITG modes [6]. In this case, sufficiently close to the unstable Alfvén continuum accumulation point, $D_M = \Lambda^2 + i\Lambda\delta W_f$, where δW_f is the MHD potential energy associated with the mode and Λ^2 is a generalized inertia given by

$$\Lambda^2 = \frac{\omega^2}{\omega_A^2}\left(1-\frac{\omega_{*pi}}{\omega}\right) + q^2\frac{\omega\omega_{ti}}{\omega_A^2}\left[\left(1-\frac{\omega_{*ni}}{\omega}\right)F(\omega/\omega_{ti}) - \frac{\omega_{*Ti}}{\omega}G(\omega/\omega_{ti}) - \frac{N^2(\omega/\omega_{ti})}{D(\omega/\omega_{ti})}\right] \quad (23)$$

and the functions $F(x)$, $G(x)$, $N(x)$ and $D(x)$, with $x = \omega/\omega_{ti}$, $\omega_{ti} = \sqrt{2}v_{thi}/(qR_0)$, and using the plasma dispersion function $Z(x)$, are defined as [6, 14]

$$F(x) = x(x^2+3/2) + (x^4+x^2+1/2)Z(x) \quad (24a)$$

$$G(x) = x(x^4+x^2+2) + (x^6+x^4/2+x^2+3/4)Z(x) \quad (24b)$$

$$N(x) = \left(1-\frac{\omega_{*ni}}{\omega}\right)\left[x + \left(1/2+x^2\right)Z(x)\right] - \frac{\omega_{*Ti}}{\omega}\left[x\left(1/2+x^2\right) + \left(1/4+x^4\right)Z(x)\right] \quad (24c)$$

$$D(x) = \left(\frac{1}{x}\right)\left(1+\frac{T_i}{T_e}\right) + \left(1-\frac{\omega_{*ni}}{\omega}\right)Z(x) - \frac{\omega_{*Ti}}{\omega}\left[x + \left(x^2-1/2\right)Z(x)\right]. \quad (24d)$$

With the new definitions

$$\tilde{\gamma}_M^2 = 2k_\vartheta^2\rho_{Li}^2\left(\frac{k_z^2 v_{thi}^2}{\omega_A^2\partial\,\mathrm{Re}\,\Lambda^2/\partial\omega_0^2}\right)\frac{\epsilon^{1/2}}{1.6q^2} \times\left(1-\frac{\omega_{*pi}}{\omega_0}-\frac{\omega_A^2}{\omega_0^2}\mathrm{Re}\,\Lambda^2\right)\left(1+\frac{\omega_A^2}{\omega_0^2}\mathrm{Re}\,\Lambda^2\right) \times\left\langle\!\!\left\langle\left|\frac{eA_0\Psi_0}{T_i}\right|^2\right\rangle\!\!\right\rangle \quad (25a)$$

$$\tilde{\Delta} = \frac{k_z^2}{2}\frac{\partial^2\delta W_f^2}{\partial k_r^2}\Big/\frac{\partial}{\partial\omega_0}\mathrm{Re}\,\Lambda^2 \quad (25b)$$

the zonal flow growth rate induced by the AITG mode is

$$\Gamma_{z,AITG} = \tilde{\gamma}_M\sqrt{1-\tilde{\Delta}^2/\tilde{\gamma}_M^2}. \quad (26)$$

As in the case of the KAW, we find a condition for effective excitation of zonal flow by the AITG mode, i.e. $\omega_0 > \omega_{*pi}$, which is the typical case for a slightly unstable AITG mode [14]. Above the threshold, the AITG driven zonal flow growth rate also scales linearly with the mode amplitude.

4. Conclusions

In the present work, we have demonstrated that zonal flows may be spontaneously generated by a variety of drift and drift-Alfvén turbulences and that above their spontaneous excitation threshold their growth rate typically scales linearly with the mode amplitudes. In the electrostatic limit, good agreement is shown between numerical results from 3-D gyrokinetic simulations of the ITG and the obtained analytic expression [9]. In the same limit, non-linear equations for mode amplitudes have recently been derived [9], and they demonstrate saturation of the linearly unstable modes via coupling to the stable envelope sidebands and, as a consequence, oscillatory behaviours in the drift wave intensity and zonal flows [9]. Similar behaviours can also be expected in the general electromagnetic case, which will be analysed in the near future.

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